

## ON PROPERTIES OF THE CATEGORY OF UNPOINTED SPACES

ABDULSATAR JMAH AL-JUBURIE<sup>1\*</sup>

**Abstract:** Let  $T_*$  the category of unpointed spaces. We introduce some results on category theory. We construct and proof some propositions and examples for the category of  $T_*$ , which we have used them to prove our main theorem: Suppose that  $Y$  be a topological space. If we define two functors on  $T_*$  by  $P(X) = X \times Y$  and  $M(Z) = \text{map}(Y, Z)$  then  $P$  and  $M$  are adjoint functors.

### INTRODUCTION

Category theory takes a general view from above of mathematics. It provides a general theoretical foundation that has proved beneficial in subjects as diverse as topology, geometry and foundational mathematics. So, a topologist its might work in the category of topology spaces and continuous functions while an algebraist might work in the category of groups and homeomorphisms. In fact, algebraic topology is a branch of mathematics that employs the concept of abstract algebra to explore the space of topological concepts. The primary objective is to transform issues regarding topological spaces and continuous functions into problems regarding algebraic objects such as groups, rings, vectors spaces and their associated homeomorphisms, see for example Tom Dieck [1], Hatcher [2] and Rotman [3]. Typically, categories and functors are employed in the field of algebraic topology. The class of pointed (respectively unpointed) spaces and the exponential law are significant objects in both category theory and algebraic topology at the same time. In our research, we will concentrate on the class of spaces that are unpointed, which is written  $T_*$ . In Section 2, we discuss the different types of categories and functions with several examples. In Section 3, we explain the concept of homotopy in terms of several useful properties and examples that are associated with our work, see examples Awodey [4] and Rotman [5]. In Section 4, we discuss the characteristics of unpointed spaces in detail. Additionally, we demonstrate some propositions that facilitate our objectives. In reality, this Section contains the proof of our primary theorem. (Theorem 4.7).

### CATEGORIES AND FUNCTORS

In this section, we'll discuss the different categories and functions with examples, additional details are available in Adnan, et al. [6], Hatcher [2], Simmons [7] and Strom [8].

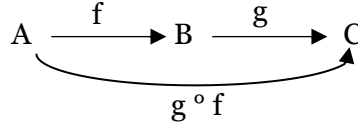
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<sup>1</sup> University of DiyalaIraq. Email: [aaljuburie@gmail.com](mailto:aaljuburie@gmail.com)

\*Correspondence: [aaljuburie@gmail.com](mailto:aaljuburie@gmail.com)

**Definition 2.1.** A **category**  $\mathcal{C}$  consists of a collection of **objects**  $Ob(\mathcal{C})$  and for every  $A, B \in Ob(\mathcal{C})$ , a set  $Mor_{\mathcal{C}}(A, B)$ , called the set of **morphisms** (arrows) from  $A$  to  $B$ . These the set of following conditions:

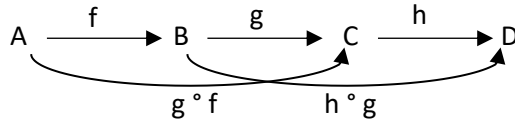
1. **Composition:** for any three objects  $A, B, C \in Ob(\mathcal{C})$ , for each pair of morphisms,  $f \in Mor_{\mathcal{C}}(A, B)$  and  $g \in Mor_{\mathcal{C}}(B, C)$ , there exists a morphisms  $g \circ f \in Mor_{\mathcal{C}}(A, C)$ . i.e.



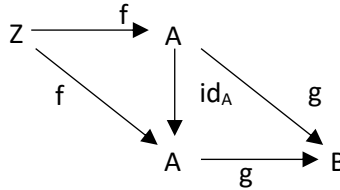
2. **Associativity:** the composition operation is associative:  $(h \circ g) \circ f = h \circ (g \circ f)$ , i.e., if

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

is a sequence of morphisms, then we can compose them as,



3. **Identity:** for any objects  $A \in Ob(\mathcal{C})$  there exist  $id_A \in Mor_{\mathcal{C}}(A, A)$  such,  $id_A \circ f = f$  for  $f \in Mor_{\mathcal{C}}(Z, A)$  and  $g \circ id_A = g$  for  $g \in Mor_{\mathcal{C}}(A, B)$ . i.e.



**Definition 2.2.** Given two categories  $\mathcal{C}$  and  $\mathcal{D}$ , a **covariant functor**  $F : \mathcal{C} \rightarrow \mathcal{D}$  dose two things:

- (i) associates to every  $A \in Ob(\mathcal{C})$ , an object  $F(A) \in Ob(\mathcal{D})$  and
- (ii) for every  $f \in Mor_{\mathcal{C}}(A, B)$ , a morphism,  $F(f) \in Mor_{\mathcal{D}}(F(A), F(B))$ , [ or :  $F(f) \in Mor_{\mathcal{C}}(F(B), F(A))$  and this is a **contravariant functor**].

These must satisfy the following conditions:

- a)  $F(g \circ f) = F(g) \circ F(f)$
- b)  $F(id_A) = id_{F(A)}$  for every  $A \in Ob(\mathcal{C})$ .

**Proposition 2.3** [3] Let  $\mathcal{C}$  be a category and let  $A, B \in ob(\mathcal{C})$ .

1. For  $f: X \rightarrow Y$ , write  $f^*: mor_{\mathcal{C}}(Y, B) \rightarrow mor_{\mathcal{C}}(X, B)$  for the function  $f^*: g \mapsto$

$g \circ f$ . Then the rules

$F(X) = \text{mor}_{\mathcal{C}}(X, B)$  and  $F(f) = f^* : F(Y) \rightarrow F(X)$  define a contravariant functor from  $\mathcal{C}$  to **Sets**.

2. For  $f: X \rightarrow Y$ , write  $f_* : \text{mor}_{\mathcal{C}}(A, X) \rightarrow \text{mor}_{\mathcal{C}}(A, Y)$  for the function

$f_* : g \mapsto f \circ g$ . Then the rules

$G(X) = \text{mor}_{\mathcal{C}}(A, X)$  and  $G(f) = f_* : G(X) \rightarrow G(Y)$  define a covariant functor from  $\mathcal{C}$  to **Sets**.

**Definition 2.4** Given two functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  (both covariant or both contravariant). A **natural transformation** is a rule  $\Phi$  mapping:

$\Phi_X : F(X) \rightarrow G(X)$  for all objects  $X \in \text{ob}(\mathcal{C})$  such that if  $f : X \rightarrow Y$  is a morphism in  $\mathcal{C}$ , the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\Phi_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\Phi_Y} & G(Y) \end{array}$$

is commutes.

**Example 2.5** Suppose that  $\phi : A \rightarrow B$  be a morphism in  $\mathcal{C}$ . If we define two functors  $\mathcal{C} \rightarrow \text{sets}$  by the rules  $F(X) = \text{mor}_{\mathcal{C}}(X, A)$  and  $G(X) = \text{mor}_{\mathcal{C}}(X, B)$ , then  $\Phi_X = \phi_* : F \rightarrow G$  is a natural transformation.

**Proof:** Let  $\phi : A \rightarrow B$  be a morphism in  $\mathcal{C}$

Let  $F : \mathcal{C} \rightarrow \text{sets}$  such that  $F(X) = \text{mor}_{\mathcal{C}}(X, A)$

Let  $G : \mathcal{C} \rightarrow \text{sets}$  such that  $G(X) = \text{mor}_{\mathcal{C}}(X, B)$

Now, Let  $f : Y \rightarrow X$  be any morphism in  $\mathcal{C}$ .

$\Phi_X = \phi_* : F(X) \rightarrow G(X)$ . So, we get the following diagram:

$$\begin{array}{ccc} F(X) = \text{mor}_{\mathcal{C}}(X, A) & \xrightarrow{\Phi_X = \phi_*} & G(X) = \text{mor}_{\mathcal{C}}(X, B) \\ F(f) = f^* \downarrow & & \downarrow G(f) = f^* \\ F(Y) = \text{mor}_{\mathcal{C}}(Y, A) & \xrightarrow{\Phi_Y = \phi_*} & G(Y) = \text{mor}_{\mathcal{C}}(Y, B) \end{array}$$

If  $g \in \text{mor}_{\mathcal{C}}(X, A)$  then by Proposition (2.3) part (a) we get that,  $f^* : g \mapsto g \circ f \in \text{mor}_{\mathcal{C}}(Y, A)$ . Also, by Proposition (2.3) part (b) we get,  $\Phi_X = \phi_* : g \mapsto \phi \circ g \in \text{mor}_{\mathcal{C}}(X, B)$ .

Similarly, we get  $\Phi_Y = \phi_* : g \circ f \mapsto \phi \circ (g \circ f) \in \text{mor}_{\mathcal{C}}(Y, B)$  and  $G(f) = f^* : \phi \circ g \mapsto (\phi \circ g) \circ f = \phi \circ (g \circ f) \in \text{mor}_{\mathcal{C}}(Y, B)$  (associative). So we get that:

$G(f) \circ \Phi_X : g \mapsto \phi \circ (g \circ f)$  and  $\Phi_Y \circ F(f) : g \mapsto \phi \circ (g \circ f)$

So the diagram is commutative. Thus,  $\Phi_X = \phi_* : F \rightarrow G$  is a natural transformation.

**Definition 2.6** Given two functors  $R : \mathcal{C} \rightarrow \mathcal{D}$  and  $L : \mathcal{D} \rightarrow \mathcal{C}$  (covariant) functors. Then for  $X \in \mathcal{D}$  and  $Y \in \mathcal{C}$ , we can define  $\text{mor}_{\mathcal{C}}(L(X), Y)$  and  $\text{mor}_{\mathcal{D}}(X, R(Y))$ , which defines functors  $M, N : \mathcal{D} \times \mathcal{C} \rightarrow \text{Sets}$  which are contravariant in the first coordinate and covariant in the second. The functors  $R, L$  are **adjoint functors** if there is a natural isomorphism between the two functors  $M$  and  $N$  ( $\Phi : M \rightarrow N$ ). That is, for each  $X \in \mathcal{D}$  and  $Y \in \mathcal{C}$ , there is an equivalenc

$$\Phi_{X,Y} : \text{mor}_{\mathcal{C}}(L(X), Y) \rightarrow \text{mor}_{\mathcal{D}}(X, R(Y)).$$

## HOMOTOPY

As we know one description of the topology is the study of all continuous maps between topological spaces. But, this is a huge problem, since there are unaccountably many maps from one space to another. Homotopy theory will solve or simplify this problem, because in homotopy we change the continuous maps by classes of continuous functions. In this section we will give the definition of the homotopy with some properties and examples, for more information see for example Abbasirad [9], Al-Juburie [10], AL-Juburie [11], Aguilar, et al. [12], Hatcher [2] and Rotman [3].

**Definition 3.1.** A **pointed space** is a topological space  $X$  with a distinguished basepoint  $x_0$  in  $X$ . Maps of pointed spaces (based maps) are continuous maps preserving basepoints, i.e. a continuous map  $f : X \rightarrow Y$  such that  $f(x_0) = y_0$ . This is usually denoted  $f(X, x_0) = (Y, y_0)$ .

**Definition 3.2.** Let  $X, Y$  be two spaces and  $I$  be the unit interval  $\{t \mid 0 \leq t \leq 1\}$ . Two maps  $f, g : X \rightarrow Y$  are called **homotopic** (written:  $f \simeq g$ ) if there exists a continuous  $\phi : X \times I \rightarrow Y$  such that  $\phi(x, 0) = f(x)$  and  $\phi(x, 1) = g(x)$  for each  $x \in X$ .

If  $g = \text{const}_y$  is a constant map (i.e., there is  $y \in Y$  such that  $g(x) = y$  for all  $x \in X$ ). Then we say that  $f$  is **null-homotopic**.

**Theorem 3.3.** The relation  $f \simeq g$  is an **equivalence relation** on the set  $\text{Hom}(X, Y)$  of continuous maps from  $X$  to  $Y$ .

**Proof:** See [2].

**Proposition 3.4** Let  $f : A \rightarrow B$  be a morphism in the category  $\mathcal{C}$ . If the induced map  $f_* : \text{mor}_{\mathcal{C}}(X, A) \rightarrow \text{mor}_{\mathcal{C}}(X, B)$  is a bijection for every  $X$ , then  $f$  is an equivalence.

**Proof:** We have  $f_* : \text{mor}_{\mathcal{C}}(X, A) \rightarrow \text{mor}_{\mathcal{C}}(X, B)$  is a bijection for every  $X$ .

Let  $X = B$ . So, we get  $f_* : \text{mor}_{\mathcal{C}}(B, A) \rightarrow \text{mor}_{\mathcal{C}}(B, B)$ . We have  $f : A \rightarrow B$ . Now, since  $f_*$  is a bijection then there exist some  $g \in \text{mor}_{\mathcal{C}}(B, A)$  such that  $f \circ g = \text{id}_B$ .

So,  $f_* : g \mapsto f \circ g = \text{id}_B$ ,  $f \circ g \in \text{mor}_{\mathcal{C}}(B, B)$ . Hence,  $g$  is a right inverse of  $f$ .

Now, Let  $X = A$ , so we get,  $f_* : \text{mor}_{\mathcal{C}}(A, A) \rightarrow \text{mor}_{\mathcal{C}}(A, B)$  which is a bijection. We have  $f : A \rightarrow B$  and  $g : B \rightarrow A$ . So,  $g \circ f : A \rightarrow A$

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & A \\ & \searrow & & \nearrow & \\ & & g \circ f & & \end{array}$$

So,  $g \circ f \in \text{mor}_{\mathcal{C}}(A, A)$ .

Now,  $f_*(g \circ f) = f \circ (g \circ f)$  (by Def.)

$$= (f \circ g) \circ f$$

$$= \text{id}_B \circ f \text{ (since } f \circ g = \text{id}_B \text{)}$$

$$= f$$

Now, we have  $\text{id}_A \in \text{mor}_{\mathcal{C}}(A, A)$ . So,  $f_*(\text{id}_A) = f \circ \text{id}_A = f$ , but  $f_*$  is a bijection, so  $f_*$  is 1-1. Hence,  $g \circ f = \text{id}_A$ . Thus,  $f_*$  is an equivalence.

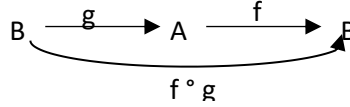
**Proposition 3.5** Let  $f : A \rightarrow B$  be a morphism in the category  $\mathcal{C}$ . If the induced map  $f_* : \text{mor}_{\mathcal{C}}(B, X) \rightarrow \text{mor}_{\mathcal{C}}(A, X)$  is a bijection for every  $X$ , then  $f$  is an equivalence.

**Proof:** We have  $f^* : \text{mor}_c(B, X) \rightarrow \text{mor}_c(A, X)$ .

Let  $X = A$ . So,  $f_* : \text{mor}_c(B, A) \rightarrow \text{mor}_c(A, A)$ . We have  $f : A \rightarrow B$ . Now, since  $f^*$  is bijection then there exist  $g \in \text{mor}_c(B, A)$  such that  $g \circ f = \text{id}_A$ . So,  $f^* : g \rightarrow g \circ f, g \circ f \in \text{mor}_c(A, A)$ . So,  $g$  is a left inverse of  $f$ .

Now, Let  $X = B$ . So,  $f^* : \text{mor}_c(B, B) \rightarrow \text{mor}_c(A, B)$ . We have  $\text{id}_B \in \text{mor}_c(B, B)$ .

So,  $f^*(\text{id}_B) = f \circ \text{id}_B = f$  (by Def).  
 $g : B \rightarrow A$ , same  $g$  above.



Now,  $f^*((f \circ g)) = (f \circ g) \circ f = f \circ (g \circ f) = f \circ \text{id}_A = f$ . But,  $f^*$  is 1-1.

So,  $f \circ g = \text{id}_B$ . Hence,  $f^*$  is equivalence.

### CATEGORY OF UNPOINTED SPACES

**Definition 4.1** Suppose that  $X$  and  $Y$  are two topological spaces.

Let  $\text{map}(X, Y) = \{f : X \rightarrow Y \mid f \text{ is continuous}\}$  the set of all continuous maps from  $X$  to  $Y$ . The **evaluation map** is defined as follows,  $@ : \text{map}(X, Y) \times X \rightarrow Y$  which is given by the formula  $(f, x) \mapsto f(x)$ .

**Remark 4.2** We will use the following notation:

1.  $T_o$  will denote the category of unpointed spaces,
2.  $T_*$  is the category of pointed spaces, and
3.  $T$  is either one.

**Theorem 4.3** [8] There is a category  $T_o$  whose objects are topological spaces (but not all topological spaces), which contains every locally compact space, and which has the following properties for any  $X, Y, Z \in T_o$ .

- a)  $\text{map}(X, Y) \in T_o$ .
- b) The evaluation  $\text{map } @ : \text{map}(X, Y) \times X \rightarrow Y$  is continuous.
- c) The composition  $\text{map} \circ : \text{map}(X, Y) \times \text{map}(Y, Z) \rightarrow \text{map}(X, Z)$  is continuous.
- d) The function  $\alpha : \text{map}(X \times Y, Z) \rightarrow \text{map}(Y, Z)$  is a homeomorphism.
- e) The colimit of any diagram of space in  $T_o$  is again a space in  $T_o$ .
- f) The limit of any diagram of spaces in  $T$  is again a space in  $T_o$ .
- g) The forgetful functor  $F : T_o \rightarrow \text{Sets}$  respects colimits.

**Remark 4.4** The  $\text{map}(X \times Y, Z) \cong \text{map}(X, \text{map}(Y, Z))$  is known as the exponential law.

**Proposition 4.5** Suppose that  $A, X, Y \in T_o$  and  $A \subseteq X$ . Then  $X/A \in T_o$  and  $X \times Y \in T_o$ .

**Proof:** First we will show  $X/A \in T_o$ . Let  $A, X \in T_o$  and  $A \subseteq X$ . Let  $X/R_A$  = the set of equivalence classes, which is the space  $X$  with  $A$  identified to a point  $[A]$ , and is written  $X/A$ .

Let  $P : X \rightarrow X/A$  is a surjective.

The identification topology in  $X/A$  determined by  $T = \{U \subset X/A \mid P^{-1}(U) \text{ is open in } X\}$

- 1)  $P(X) = X/R_A$   
 $\Rightarrow X = P^{-1}(X/R_A)$  is open in  $X$ , so  $X/R_A \in T$ .

Also,  $\emptyset \in T$ .

- 2) Let  $\cup_{i \in I} U_i \subset X/A$   
 $\Rightarrow P^{-1}(\cup_{i \in I} U_i) = \cup_{i \in I} P^{-1}(U_i)$  (property)  
 $\Rightarrow \cup_{i \in I} P^{-1}(U_i)$  is open in  $X$ .

So,  $U_i \in T$ .

3) Let  $\bigcap_{i=1}^n U_i \subset X/A$

$\Rightarrow P^{-1}(\bigcap_{i=1}^n U_i) = \bigcap_{i=1}^n P^{-1}(U_i)$  is open in  $X$ . So,  $U_i \in T$ . Hence,  $X/A$  is a topology. So,  $X/A \in T_o$ .

Now we will show that  $X \times Y \in T_o$ . We define that:

$T_{X \times Y} = \{U \subset X \times Y \mid \forall (x, y) \in U \exists U_1 \in T_X, U_2 \in T_Y \text{ such that } (x, y) \in U_1 \times U_2 \subset U \subset X \times Y\}$ . Now,

1. Since  $X \times Y \subset X \times Y \mid \forall (x, y) \in X \times Y \exists x \in T_X, y \in T_Y \text{ such that } (x, y) \in X \times Y$ .  
So,  $X \times Y \in T_{X \times Y}$ . Also,  $\phi \in T_{X \times Y}$ .

2. Let  $\bigcup_{i \in I} U_i \subset X \times Y$

Let  $(x, y) \in \bigcup_{i \in I} U_i \Rightarrow (x, y) \in U_i$  for some  $i$ .

$\Rightarrow \exists U_1 \in T_X, U_2 \in T_Y$  such that  $(x, y) \in U_1 \times U_2 \subset U \subset X \times Y$ .

3. Let  $\bigcap_{i=1}^n V_i \subset X \times Y$

Let  $(x, y) \in \bigcap_{i=1}^n V_i \Rightarrow (x, y) \in V_i$  for some  $i$ .

$\Rightarrow \exists U_i \in T_X$  s.t.  $x \in U_i$  and  $\exists U'_i \in T_Y$  s.t.  $y \in U'_i$

$\Rightarrow x \in \bigcap_{i=1}^n U_i = A \in T_X$  and  $y \in \bigcap_{i=1}^n U'_i = A' \in T_Y$

$\Rightarrow (x, y) \in A \times A' \subset \bigcap V_i \subset X \times Y$ .

Hence,  $X \times Y$  is a topology. So,  $X \times Y \in T_o$ .

**Proposition 4.6** Suppose that  $F, G: T_o \rightarrow T_o$  are two functors, defined by the formula:

$F(X) = \text{map}(X \times Y, Z)$  and  $G(X) = \text{map}(X, \text{map}(Y, Z))$ . Then the exponential law defines a natural equivalence between these two functors.

**Proof:** We define the two functors  $F, G: T_o \rightarrow T_o$  by the rules,  $F(X) = \text{map}(X \times Y \times Z)$  and  $G(X) = \text{map}(X, \text{map}(Y, Z))$ . Let  $\Phi: F(X) \rightarrow G(X)$ . So,  $\Phi: \text{map}(X \times Y, Z) \rightarrow \text{map}(X, \text{map}(Y, Z))$ . We need to show that  $\Phi$  defines a natural equivalence. Now, corresponding to an  $f \in \text{map}(X \times Y, Z)$  then  $f: X \times Y \rightarrow Z$  continuous such that  $f((x, y)) = f_x(y)$ . We define  $\Phi f: X \rightarrow \text{map}(Y, Z)$  by  $((\Phi f)_x)_y = f(x, y)$ . To show this for each  $x$ ,  $(\Phi f)_x$  is continuous from  $Y$  to  $Z$ :

Suppose that  $(\Phi f)_x$  takes  $y_0 \in Y$  into the open set  $U$  of  $Z$ . Then  $f(x, y_0) \in U$ , now since  $f$  is continuous, then this gives an open set  $V$  of  $Y$  containing  $y_0$  such that  $f(X \times V) \subset U$ . Therefore,  $(\Phi f)_x$  maps  $V$  into  $U$ . Now, we need to prove that:

If  $\in \text{map}(X, Y, Z)$ , then  $\Phi f: X \rightarrow \text{map}(Y, Z)$  is continuous .....  $\odot$

Let  $W(B, U)$  be the subbasis open set of  $\text{map}(Y, Z)$ , and suppose that,  $(\Phi f)_{x_0} \in W(B, U)$  then  $f(B \times x_0) \subset U$ . Since,  $U$  is open and  $B$  is compact, then there is a neighborhood  $N$  of  $x_0$  such that  $f(B \times N) \subset U$ . Implies that,  $(\Phi f)_N \subset W(B, U)$  and we are done i.e.,  $\Phi f$  is continuous. Now, to prove that  $\Phi$  is continuous:

We will start with the continuity of the evaluation mapping by rearranged as,

$$@ : X \times Y \times \text{map}(X \times Y, Z) \rightarrow Z$$

We will apply  $\odot$  with  $X$  replaced by  $\times \text{map}(X \times Y, Z)$ , then we will find that  $\Phi @: X \times \text{map}(X \times Y, Z) \rightarrow \text{map}(Y, Z)$  is continuous. Now, we apply  $\odot$  again, with  $X$  replaced by  $\text{map}(X \times Y, Z)$ ,  $Y$  by  $X$  and  $Z$  by  $\text{map}(Y, Z)$ , then we get that:

$\Phi(\Phi @): \text{map}(X \times Y, Z) \rightarrow \text{map}(X, \text{map}(Y, Z))$  which is continuous. Note that,  $\Phi(\Phi @)$  coincides  $\Phi$ . We need to show that  $\Phi$  has a continuous inverse:

Let  $@: X \times \text{map}(X, \text{map}(Y, Z)) \rightarrow \text{map}(Y, Z)$  and  $@': Y \times \text{map}(Z, Y) \rightarrow Z$  are evaluation mappings. Now, the composition  $@': (1 \times @): Y \times X \times \text{map}(X, \text{map}(Y, Z)) \rightarrow Z$  is continuous (Theorem 4.3, part (C)), the composition map is

continuous).

Now, we applying  $\otimes$ , with  $X$  replace by  $\text{map}(X, \text{map}(Y, Z))$ ,  $Y$  by  $Y \times X$ , and  $Z$  by  $Z$ , we will get that:

$\Phi(@'(1 \times @)) : \text{map}(X, \text{map}(Y, Z)) \rightarrow \text{map}(Y \times X, Z)$  is defined and continuous.

So, it is verified that  $\Phi(@'(1 \times @))$  is the inverse of  $\Phi$ . Hence,  $\Phi$  is equivalence.

Now we will use the exponential law to define a natural transformation:

Let  $F, G : T_0 \rightarrow T_0$  be two functors which are define by:

$F(X) = \text{map}(X \times Y, Z)$  and  $G(X) = \text{map}(X, \text{map}(Y, Z))$

From above we get that,

$$\text{map}(X \times Y, Z) \cong \text{map}(X, \text{map}(Y, Z)). \text{ So,}$$

$$F(X) = \text{map}(X, Y, Z) \cong \text{map}(X, \text{map}(Y, Z))$$

Let  $g : X \rightarrow X'$  in  $T_0$  be any morphism. We know  $\text{map}(X, Y) = \text{mor}_{\text{Top}}(X, Y) = \{f : X \rightarrow Y, \text{continuous}\}$ . So we get the following diagram:

$$\begin{array}{ccc} F(X) = \text{map}_{\text{Top}}(X, \text{map}(Y, Z)) & \xrightarrow{\Phi_X = \phi_*} & G(X) = \text{map}_{\text{Top}}(X, \text{map}(Y, Z)) \\ F(g) = g^* \downarrow & & \downarrow G(g) = g^* \\ F(X') = \text{map}_{\text{Top}}(X', \text{map}(Y, Z)) & \xrightarrow{\Phi_{X'} = \phi_*} & G(X') = \text{map}_{\text{Top}}(X', \text{map}(Y, Z)) \end{array}$$

Now, If  $\Phi f \in \text{map}_{\text{Top}}(X, \text{map}(Y, Z)) \mapsto g^*(\Phi f) = (\phi f) \circ g \mapsto \phi_*((\phi f) \circ g) = \phi \circ (\Phi f \circ g)$ .

Also, If  $\Phi f \in \text{map}_{\text{Top}}(X, \text{map}(Y, Z)) \mapsto \phi_*(\Phi f) = \phi \circ (\Phi f) \mapsto g^*(\phi \circ (\Phi f)) = (\phi \circ (\Phi f)) \circ g = \phi \circ (\Phi f \circ g)$ .

By Proposition 2.3 part (i), we get  $F(g) = g^* : \Phi f \mapsto \Phi f \circ g$

Also, by Proposition 2.3 part (ii), we get  $\Phi_X = \phi_* : \Phi f \mapsto \phi \circ \Phi f$

Similarly, we get:

$$\begin{aligned} \Phi_Y &= \phi_* : (\Phi f) \circ g \mapsto \phi \circ (\Phi f \circ g), \text{ and} \\ G(g) &= g^* : \phi \circ \Phi f \mapsto (\phi \circ \Phi f) \circ g = \phi \circ (\Phi f \circ g) \text{ (associative).} \end{aligned}$$

So, we get,

$G(g) \circ \Phi_X : \Phi f \mapsto \phi \circ (\Phi f \circ g)$  and  $\Phi_Y \circ F(g) : \Phi f \mapsto \phi \circ (\Phi f \circ g)$ .

Hence, the diagram is commutative. Thus,  $\Phi_X = \phi_* : F(X) \rightarrow G(X)$  is a natural transformation.

The next theorem is the main result in our work.

**Theorem 4.7 (Main theorem)** Suppose that  $Y$  be a topological space. If we define two functors on  $T_0$  by  $P(X) = X \times Y$  and  $M(Z) = \text{map}(Y, Z)$  then  $P$  and  $M$  are adjoint functors.

**Proof:** We define two functors  $P, M : T_0 \rightarrow T_0$  by:

$$P(X) = X \times Y \text{ and } M(Z) = \text{map}(Y, Z)$$

are two (covariant) functors. Then for  $X, Z \in T_0$  we have:

$$mor_{Top}(P(X), Z) \text{ and } mor_{Top}(X, M(Z))$$

Which are define the following functors,

$$(X, Z) \xrightarrow{K} mor_{Top}(P(X), Z) \text{ and } (X, Z) \xrightarrow{N} mor_{Top}(X, M(Z))$$

We need to prove there is a natural isomorphism between the two functors  $K$  and  $N$ , i.e.,  $\Phi : K \rightarrow N$ .

$$\begin{aligned} \Phi_{X,Z} : mor_{Top}(P(X), Z) &\rightarrow mor_{Top}(X, M(Z)) \\ \text{So, } \Phi_{X,Z} : mor_{Top}(X \times Y, Z) &\rightarrow mor_{Top}(X, map(Y, Z)) \end{aligned}$$

but, from Proposition 4.6 we prove that  $\Phi$  is a natural isomorphism. Hence,  $P$  and  $M$  are adjoint functors.

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