

UNION-CLOSED FAMILIES OF SETS AND THE FRANKL CONJECTURE

JOHN L. PFALTZ^{1*}

Abstract: The “union-closed” conjecture, sometimes attributed to Peter Frankl, dates from 1979. It asserts that in any finite, union-closed family, $\mathcal{F}$, of sets, there is some element x that is a member of at least half the sets in $\mathcal{F}$. It seems simple, but to date no one has been able to either prove or disprove it. In this article we prove it to be true. In the course of this proof we show that even seemingly randomly generated union-closed families of sets have a rich auxiliary structure, and that this structure is determined by the largest sets in $\mathcal{F}$, rather than the smallest sets, often called “atoms”. We will show that all union closed families fall into two distinct classes, and that two different proof techniques are needed to prove that the conjecture is indeed true.

A family of sets, $\mathcal{F}$, is said to be **union-closed** if $X, Y \in \mathcal{F}$ implies $Z = X \cup Y \in \mathcal{F}$. All sets, X, Y, Z are sets over some finite universe U of elements. In the remainder of this paper, $\mathcal{F}$, which we will refer to as a U-C family, is always assumed to be union-closed.

Conjecture: Given a non-empty, finite, union-closed family, $\mathcal{F} \neq \{\emptyset\}$, over a set of elements, U , there exists some element $x \in U$ such that x is an element in (member of) at least half the sets of $\mathcal{F}$.

This conjecture, which has been attributed to Peter Frankl in 1979, appears to be absurdly simple. Yet, Bruhn and Schaudt observe that “despite its apparent simplicity the union-closed sets conjecture remains wide open. This is certainly not for lack of interest — there are about 50 articles dedicated to the conjecture as well as several websites” [1].

We can rewrite the conclusion of this conjecture as: there exists a set $\{x\} \subset U$ such that

$$|\{Y \in \mathcal{F} : \{x\} \subseteq Y\}| \geq |\{Y \in \mathcal{F} : \{x\} \not\subseteq Y\}|, \quad (1)$$

since we can always consider the element x as a singleton set. $\{x\}$ need not be a set in $\mathcal{F}$, and seldom is. We say such elements x , or sets $\{x\}$, are “abundant”. Bruhn and Schaudt explain that “we do not know where to expect an abundant element” [1]. In this paper we show that one looks for an abundant set, or element, in different places depending on the internal structure of $\mathcal{F}$. In Section 3, Lemmas 2.1 and 3.1 culminate in Proposition 3.2 to establish that it is the complement of one of the largest sets that must be abundant. In Section 4, Proposition 4.4 details where among the smallest sets one can find an abundant set.

This conjecture can be regarded as the “holy grail” of union-closed (U-C) families of sets.

¹ Department of Computer Science, University of Virginia, United States. Email: jl@virginia.edu

*Correspondence: jl@virginia.edu

1. FAMILIES OF SETS CLOSED UNDER UNION

A set Z is said to **cover** a set X with respect to $\mathcal{F}$ if $X \subset Z$ but if $X \subseteq Y \subseteq Z$, where $Y \in \mathcal{F}$, then either $X = Y$ or $Y = Z$, i.e. there can be no set $Y \in \mathcal{F}$ “between” X and Z . If a set $Y \in \mathcal{F}$ covers the empty set, $\emptyset$, we call Y an **atom** of $\mathcal{F}$ and denote it by A_i . The collection of all sets $\{A_i\}$ covering $\emptyset$, we denote by $\mathcal{A}$. Conversely, those sets Z_i covered by U , we call **co-atoms** and let $\mathcal{Z}$ denote this collection $\{Z_i\}$.

By $\bar{X}$ we mean $U \sim X$, or the **complement** of X . The dual nature of complements, such as (a) $X \subseteq Y$ if and only if $\bar{Y} \subseteq \bar{X}$, and (b) $Z = X \cup Y$ if and only if $\bar{Z} = \bar{X} \cap \bar{Y}$, is well known and easily demonstrated.

A non-empty set Z in an union-closed family $\mathcal{F}$ is said to be **reducible**, if it is the union of two distinct, non-empty sets of $\mathcal{F}$. It is **irreducible** if there do not exist two such sets, in which case there are only two possible situations:

(1) Z covers $\emptyset$ (so Z is an atom); or

(2) Z consists of a set $X \in \mathcal{F}$ and a set $[Y] = Z \sim X$, where $[Y] \notin \mathcal{F}$ ²

Readily, a non-empty set Z , that is not an atom, is irreducible if and only if it covers a single set $Y \in \mathcal{F}$.

Lemma 1.1

Let $\mathcal{F}$ be a U-C family with $Y \in \mathcal{F}$. $\mathcal{F}' = \mathcal{F} \sim Y$ is a U-C family if and only if Y is irreducible.

Proof: If Y is reducible, $Y = X \cup V$ where $X, V \in \mathcal{F}$ implying $\mathcal{F} \sim Y$ will not be union closed; so Y must be irreducible. Conversely, any irreducible set can be deleted from a U-C family to yield another U-C family because it covers at most a single set. $\square$

It is curious that it is essentially trivial to obtain another U-C family, $\mathcal{F}'$, by removing only one irreducible set; but it is extremely difficult to add just a single set and still be union-closed.

By an **interval** in $\mathcal{F}$ we mean $\langle X, Z \rangle = \{Y \in \mathcal{F} : X \subset Y \subset Z\}$. Note that the bounding sets X and Z are not elements of the interval and need not be in $\mathcal{F}$.

Lemma 1.2

Let $Z = X \cup Y$ be reducible. Then $Z \sim X \subseteq Y$ and $Z \sim Y \subseteq X$ or equivalently $Z \cap \bar{X} \subseteq Y$ and $Z \cap \bar{Y} \subseteq X$.

Proof: Suppose not. Then there exists a non-empty $W \subset Z$, $W \not\subseteq X$, $W \not\subseteq Y$ implying $X \cup Y \subseteq W \subset Z$ contradicting the assumption that $Z = X \cup Y$. The equivalent expressions follow from the observation that $Z \sim X \equiv Z \cap \bar{X}$. $\square$

Figure 1 (a) and (b) illustrates the union of two intervals, $\langle X, Z \rangle$ and $\langle Y, Z \rangle$ where in each case, $X = \{123\}$, $Y = \{42\}$ and $Z = \{12356\}$. Lines highlight set containment.

² We use square brackets, [...] as in $X \cup [Y]$, to signal that the enclosed set is not a set of $\mathcal{F}$. Similarly, we use $[x]$ when $\{x\} \notin \mathcal{F}$.

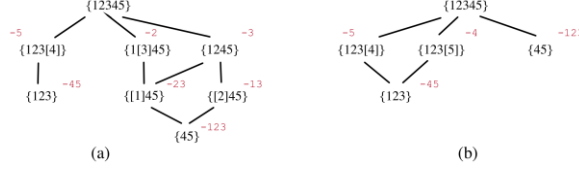


Figure 1: Two configurations of $\langle V, Z \rangle \cup \langle W, Z \rangle$ where $V = \{123\}$ and $W = \{45\}$.

Above each set covered by $Z = \{12356\}$ we have indicated its complement relative to Z as $-k$, illustrating Lemma 1.2. In our figures, sets delimited by braces $\{\dots\}$ are presumed to be members of $\mathcal{F}$; sets delimited by square brackets $[\dots]$ are not members of $\mathcal{F}$. Union-irreducible sets, such as $\{123[4]\}$ in Figure 1 employ both. (Our figures are used only to illustrate various configurations of U-C families; not to suggest the truth of the conjecture. Exhaustive computer programs have shown that the conjecture is true for all families, $\mathcal{F}$, for which $|U| \leq 12$ [2] and our figures are much smaller than that!) These two configurations clearly demonstrate the extreme variation possible between intervals with the same bounds, and that $|X, Z|$ is independent of $|X|$.

2. CO-ATOMS

Let Z_k be a co-atom, and recall that $\bar{Z}_k = U \sim Z_k$. Clearly, for all $Y \in \langle \emptyset, Z_k \rangle$, $\bar{Z}_k \not\subseteq Y$ since $\bar{Z}_k \not\subseteq Z_k$. The following lemma echoes Lemma 1.2.

Lemma 2.1

Let $Y \in \mathcal{F}$, If $Y \not\subseteq Z_k$ then $\bar{Z}_k \subseteq Y$. Or equivalently, $Y \notin \langle \emptyset, Z_k \rangle$ then $Y \in \langle Z_k, U \rangle$.

Proof: $Y \not\subseteq Z_k$ implies $Z_k \subseteq Y \cup Z_k \subseteq U$. But U covers Z_k , so either $Z_k = Y \cup Z_k$ or $Y \cup Z_k = U$. Since the former is impossible by assumption, $\bar{Z}_k \subseteq Y$. $\square$

This is a rather surprising lemma and worth examining further. In Figure 2, where the co-atoms are numbered left to right, we have delimited $\langle \emptyset, Z_3 \rangle = \langle \emptyset, \{12356\} \rangle$ with bolder lines and a dashed enclosure. $\bar{Z}_3 = [4]$ and we observe that for every non-empty set $Y \notin \langle \emptyset, Z_3 \rangle$, $[4] \subseteq Y$, even though $[4]$ is not a set of $\mathcal{F}$.

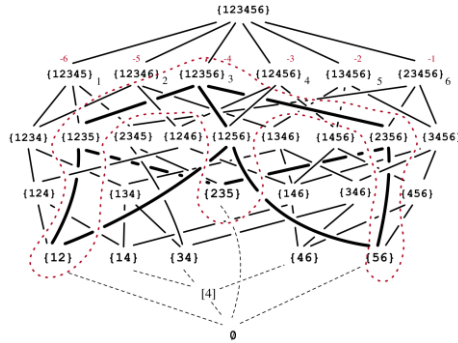


Figure 2: $\langle \emptyset, \{12356\} \rangle$, $\bar{Z}_3 = [4]$.

Another example involves Poonen's classic U-C family $[4]$, shown as Figure 3, which was created to provide a counter-example to the rather natural supposition that if

$|A_i|$ were minimal, then the members of A_i would also be members of many other sets Y . It is easy to show that any singleton set $\{a\} \in \mathcal{F}$ is “abundant” since for all $Y \in \mathcal{F}$ such that $\{a\} \not\subseteq Y$, there exists $\{a\} \cup Y \in \mathcal{F}$. This can be extended to show that if $\mathcal{F}$ has a doubleton atom, $\{ab\}$ its elements are also “abundant”. Bruhn and Schaudt [1] observe that trying to extend this approach to tripleton atoms, $\{abc\}$ fails. So, we are only concerned with U-C families for which $Y \in \mathcal{F}$ implies $|Y| \geq 3$. Poonen’s U-C family [4] shown as Figure 3, is one counter example to demonstrate this. Readily, $\{123\}$ is the minimal atom; but for all $x \in \{123\}$, $|\{Y : [x] \subseteq Y\}| = 10 < 13 = 1/2 * 26 = |\mathcal{F}|/2$. We use this U-C family to provide a second illustration of Proposition 2.1. In Figure 3 we have delimited $\langle \emptyset, Z_4 \rangle$, which is not minimal, with a dashed line and enboldened their connections. Observe that for all non-empty $Y \notin \langle \emptyset, Z_4 \rangle$, $[3] = \bar{Z}_4 \subseteq Y$. In the U-C families illustrated in Figures 2 or 3, neither $\bar{Z}_3 = [4]$ in the former nor $\bar{Z}_4 = [3]$ of the latter is a subset of $\mathcal{F}$.

Before continuing we should resolve a few details.

As observed above, we can ignore any family in which either a singleton set $\{x\}$ or doubleton $\{xy\} \in \mathcal{F}$, so in the families of interest any single abundant element, x , expressed as a set must be $[x] \notin \mathcal{F}$ and we can seek to demonstrate the inequality (2)

$$|\{Y \in \mathcal{F} : [x] \subseteq Y\}| \geq |\{Y \in \mathcal{F} : [x] \not\subseteq Y\}| \quad (2)$$

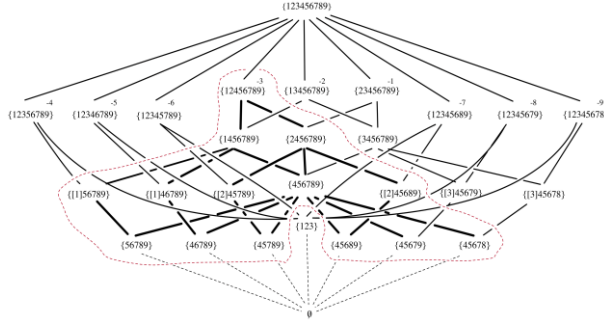


Figure 3: The U-C family of Poonen [4].

If $|\mathcal{A}| = 1$ then $A_1 \in \mathcal{A}$ is contained in every non-empty set $Y \in \mathcal{F}$, so any $[x] \subseteq A_1$ clearly satisfies (2). If $|Z| = 1$ then $\mathcal{F}' = \mathcal{F} \cup U$ where $\mathcal{F}' = \{Y : Y \subseteq Z_1\}$. If $\mathcal{F}'$ satisfies (2), then clearly $\mathcal{F}$ does as well; or equivalently if $\mathcal{F}$ does not satisfy the conjecture, then there is a smaller U-C family that does not as well. In either case, we need only concern ourselves with U-C families such that $|\mathcal{A}| > 1$ and $|Z| > 1$.

Finally, although technically U must be in $\mathcal{F}$ because it is union-closed, it is simpler to analyze this family of sets if we pretend that $\emptyset, U \notin \mathcal{F}$. This in no way changes the validity of the inequalities (1) or (2) expressing Frankl’s Conjecture since for all sets $[x]$, $[x] \subseteq U$ and $[x] \not\subseteq \emptyset$ so they cancel each other out. For the rest of this paper, $\mathcal{F}$ will denote the interval $\langle \emptyset, U \rangle$.

3. WHEN $\cap_j \{Z_j \in \mathcal{Z}\} = \emptyset$

In this section we explore those U-C families for which $\cap_j \{Z_j \in \mathcal{Z}\} = \emptyset$. Since

$\cap_j \{Z_j \in Z\} = U \sim \cup_j \{\bar{Z}_j \in Z\}$ we will assume that each $\bar{Z}_i$ denotes a single element of U . This class corresponds to the “first class” of FC(n) families explored by Vaughan [6], that is, the n -set U , together with all of its $(n-1)$ -subsets. The U-C families of Figures 2 and 3 are members of this class.

Proposition 2.1 identifies one source of many abundant sets, namely the complements of co-atoms, $\bar{Z}_k$. This will lead to Proposition 3.2 which provides a partial answer to the U-C conjecture. But, first we must establish that if $|\langle \emptyset, Z_k \rangle|$ is minimal then $|\langle \emptyset, Z_k \rangle| \leq |F|/2$, provided $\cap_j \{Z_j \in Z\} = \emptyset$. Since $|\langle \emptyset, Z_k \rangle|$ is minimal, for each $Y_i \in \langle \emptyset, Z_k \rangle$ there must be a corresponding X_i in each of the other $\langle \emptyset, Z_j \rangle$, so this seems intuitively obvious. But, one can have $X_i = Y_i$ for all $\langle \emptyset, Z_j \rangle$ as shown in Figure 4 where $|\langle \emptyset, Z_1 \rangle| = |\{\{1\}2\}, \{2\}\}| = 2 > 3/2 = |F|/2$. The proviso that $\cap_j \{Z_j \in Z\} = \emptyset$ is essential.

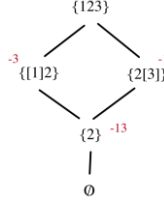


Figure 4: $\cap_j \{Z_j \in Z\} = \{2\} \neq \emptyset$.

Lemma 3.1

Let $\cap_j \{Z_j \in Z\} = \emptyset$. If $|\langle \emptyset, Z_k \rangle|$ is minimal, then $|\langle \emptyset, Z_k \rangle| \leq |F \sim \langle \emptyset, Z_k \rangle|$. We have equality if and only if for each $Y_i \in \langle \emptyset, Z_k \rangle$, there exists $X_i = \bar{Y}_i \in F \sim \langle \emptyset, Z_k \rangle$.

Proof: We run a finite induction on $|\langle \emptyset, Z_k \rangle|$. Let $|\langle \emptyset, Z_k \rangle| = 1$ with $Y_1 \in \langle \emptyset, Z_k \rangle$. Since $|\langle \emptyset, Z_k \rangle|$ is minimal, $|\langle \emptyset, Z_j \rangle| \geq 1$ for all $j \neq k$. If $\exists \langle \emptyset, Z_j \rangle$ such that $Y_1 \notin \langle \emptyset, Z_j \rangle$ then $\exists X_1 \neq Y_1$ where $X_1 \in \langle \emptyset, Z_j \rangle$ implying $X_1 \in F \sim \langle \emptyset, Z_k \rangle$. But suppose $|F \sim \langle \emptyset, Z_k \rangle| = 0 < 1$, then $Y_1 \in \langle \emptyset, Z_j \rangle$ for all j contradicting the assumption that $\cap_j \{Z_j \in Z\} = \emptyset$.

On the other hand, if $|F \sim \langle \emptyset, Z_k \rangle| = 1$ then $X_1 \in \langle \emptyset, Z_j \rangle$ for all $j \neq k$, implying $X_1 = \bar{Z}_k$.

Assume the lemma is true for $|\langle \emptyset, Z_k \rangle| < m$ and let $|\langle \emptyset, Z_k \rangle| = m$. By inductive assumption $|F \sim \langle \emptyset, Z_k \rangle| \geq m - 1$. If $|F \sim \langle \emptyset, Z_k \rangle| = m - 1$, the argument above implies $Y_m \in \cap_j \{Z_j \in Z\}$ and the same contradiction. If $|F \sim \langle \emptyset, Z_k \rangle| = m$, then $\bar{Y}_m \in \langle \emptyset, Z_j \rangle$ for all $j \neq k$. $\square$

Proposition 3.2

Let $|Z| = |U| \geq 2$ and let $\cap_j \{Z_j \in Z\} = \emptyset$. If $|\langle \emptyset, Z_k \rangle|$ is minimal then $|\{Y \in F : \bar{Z}_k \subseteq Y\}| \geq |\{Y \in F : \bar{Z}_k \not\subseteq Y\}|$

Proof: Since $F = \langle \emptyset, U \rangle$, by Lemmas 2.1 and 3.1, $|\{Y \in F : \bar{Z}_k \not\subseteq Y\}| = |\langle \emptyset, Z_k \rangle| \leq |F \sim \langle \emptyset, Z_k \rangle| = |\{Y \in F : \bar{Z}_k \subseteq Y\}|$. Reverse the inequality. $\square$

Consequently, this result resolves Frankl’s conjecture for this large class of U-C families. $[x] \subseteq \bar{Z}_k$ is an abundant set for any co-atom, Z_k , such that $|\langle \emptyset, Z_k \rangle|$ is minimal.

4. WHEN $\cap_j \{Z_j \in \mathcal{Z}\} = I \neq \emptyset$

In this section we are concerned with the structure of $\mathcal{F}$ when the intersection, I , of all the co-atoms $Z_j \in \mathcal{Z}$ is non-empty. The resulting set, I , may be a member of $\mathcal{F}$, or not. A completely different proof structure is required.

Figure 5 illustrates two representative U-C families for which $I = \cap_j \{Z_j \in \mathcal{Z}\} \neq \emptyset$. In Figure 5(a), $I = \{12\} = \{12356\} \sim \{345\} = U \sim \cup_j \{\bar{Z}_j\} \in \mathcal{F}$. In Figure 5(b), $I = [15] \notin \mathcal{F}$. In both, the interval $\langle I, U \rangle$ has been emboldened, and $\mathcal{F} \sim \langle I, U \rangle$ is denoted with thinner, but solid, lines. One expects that an abundant set will be found in I , or some subset of I . In both figures it is easy to verify that I is itself abundant.

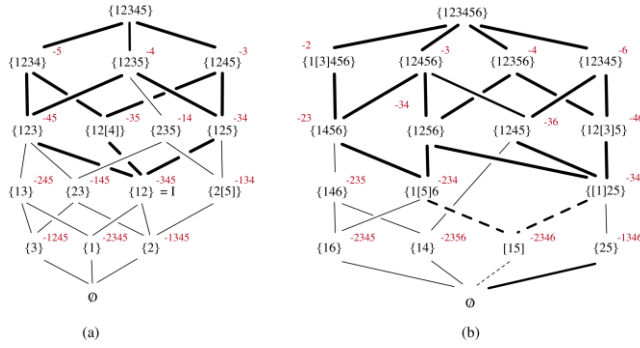


Figure 5: Two U-C families in which $I = \cap_j \{Z_j \in \mathcal{Z}\} \neq \emptyset$.

However, this need not always be the case. In Figure 6, $I = [3]$ is clearly not abundant since $|\{Y \in \mathcal{F} : [3] \subseteq Y\}| = 4 < 7 = |\{Y \in \mathcal{F} : [3] \not\subseteq Y\}|$. And readily $I = [3]$ has no subset that might be abundant.

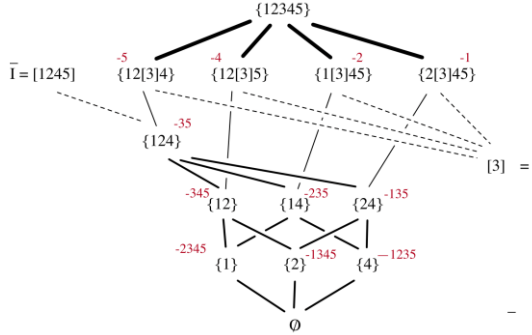


Figure 6: $I = [3]$ is not abundant.

Proposition 4.1

If $|\langle I, U \rangle| \geq |\mathcal{F} \sim \langle I, U \rangle|$ then $|\{Y \in \mathcal{F} : I \subseteq Y\}| \geq |\{Y \in \mathcal{F} : I \not\subseteq Y\}|$

Proof: Trivial $\square$

Proposition 4.1 resolves the conjecture whenever $|\langle I, U \rangle| \geq |\mathcal{F} \sim \langle I, U \rangle|$; I itself and

any subset $[x]$ will be abundant. Consequently, we need only be concerned when the number of sets not in $\langle I, U \rangle$ outnumber those in it. It is no longer assured that an abundant set $[x]$ will be contained in I ; we may find $[x] \in \mathcal{F} \sim \langle I, U \rangle$. The rest of this section is focused on the structure of $\mathcal{F} \sim \langle I, U \rangle$.

Proposition 4.2

If $\mathcal{F} \sim \langle I, U \rangle \subseteq \langle \emptyset, I \rangle$ then there exists $[x] \subseteq I$ such that $|\{Y \in \mathcal{F}: [x] \subseteq Y\}| > |\{Y \in \mathcal{F}: [x] \not\subseteq Y\}|$

Proof: $\mathcal{F}' = \langle \emptyset, I \rangle$ is a U-C family contained in $\mathcal{F}$. It is assumed that the U-C conjecture is true for $\mathcal{F}'$. Let $[x]$ be that set. Since $[x] \subseteq I, [x] \subseteq Y$ for all $Y \in \langle I, U \rangle$ and the result follows easily. $\square$

A key subset of $\mathcal{F} \sim \langle I, U \rangle$ is the interval $\langle \emptyset, \bar{I} \rangle$. $\bar{I} \notin \mathcal{F}$ since otherwise $\bar{I} \in Z$ and $\cap_j \{Z_j \in Z\} = \emptyset$, contradicting the assumed co-atom structure. (In fact, no set $Y \in \langle \bar{I}, U \rangle$ can be in $\mathcal{F}$ for the same reason). $\langle \emptyset, \bar{I} \rangle$ in Figure 6 is interesting. $\bar{I} = [1245] \notin \mathcal{F}$ and $\langle \emptyset, \bar{I} \rangle = \langle \emptyset, [1245] \rangle$ consists of the seven sets $\{\{1\}, \dots, \{124\}\}$, so $|\langle \emptyset, \bar{I} \rangle| = 7 > 4 = |\langle [3], U \rangle| = |\langle I, U \rangle|$. Because $\langle \emptyset, \bar{I} \rangle$ is U-C, $\cup_i \{Y_i \in \langle \emptyset, \bar{I} \rangle\} \subset \bar{I}$. Consequently, $\langle \emptyset, \bar{I} \rangle$ must have a unique greatest element which is an element of $\mathcal{F}$, but cannot be $\bar{I}$. We see this in Figure 6 where $\{124\} \subset [1245] = \bar{I}$.

Proposition 4.3

If $\langle \emptyset, \bar{I} \rangle$ is not empty and $\langle \emptyset, \bar{I} \rangle = \mathcal{F} \sim \langle I, U \rangle$ then there exists $[x] \subseteq \bar{I}$ satisfying the inequality (2).

Proof: Since $\langle \emptyset, \bar{I} \rangle$ is a U-C family $|\langle \emptyset, \bar{I} \rangle| < |\mathcal{F}|$ we may assume there exists an abundant $[x] \in A_1$ where $\emptyset \subset A_1 \subset \bar{I}$. Now let $Y_i \in \langle I, U \rangle, [x] \not\subseteq Y_i$. Since $\mathcal{F}$ is U-C there exists $X_i = Y_i \cup A_1$ such that $[x] \subseteq X_i$. Moreover, because I and $\bar{I}$ are disjoint, $Y_i \neq Y_j \in \langle I, U \rangle$ implies $X_i = Y_i \cup A_1 \neq Y_j \cup A_1 = X_j$. Thus $[x]$ is abundant in $\mathcal{F}$. $\square$

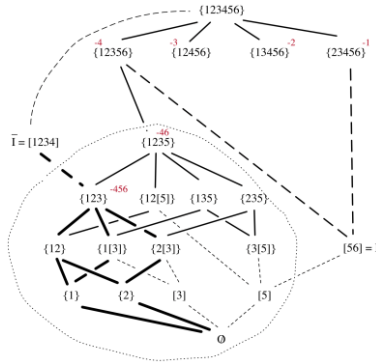


Figure 7: A schematic representation of $\langle \emptyset, \bar{I} \rangle$ contained in $\mathcal{F} \sim \langle I, U \rangle$.

Figure 7 graphically illustrates relationships between $\langle I, U \rangle$, $\mathcal{F} \sim \langle I, U \rangle$, and $\langle \emptyset, \bar{I} \rangle$. $I = [56]$ which may, or may not, be a set in $\mathcal{F}$. $\langle I, U \rangle$ is only sketched out with dashed lines; it may consist of other sets, but need not be. But, we assume that $|\langle I, U \rangle| <$

$|\mathcal{F} \sim \langle I, U \rangle|$ else Proposition 4.1 is invoked. If $I = \{56\} \in \mathcal{F}$ there would be at least 11 more sets, $\{156\}, \dots, \{12356\} \in \langle I, U \rangle$. $\mathcal{F} \sim \langle I, U \rangle$ has been circled by a dotted line and $\langle \emptyset, \bar{I} \rangle$ within it is indicated by bolder lines.

Because of Propositions 4.1, 4.2 and 4.3, the only remaining case involves $\mathcal{F} \sim \langle I, U \rangle$, where $|\mathcal{F} \sim \langle I, U \rangle| \geq |\langle I, U \rangle|$ and there exists at least one $Y \in \mathcal{F} \sim \langle I, U \rangle$ such that $Y \cap I \neq \emptyset$, yet $Y \not\subseteq I$.

Proposition 4.4

Let $\cap_j \{Z_j \in \mathcal{Z}\} = I \neq \emptyset$ and let there exist $Y \in \mathcal{F} \sim \langle I, U \rangle$ such that $Y \cap I \neq \emptyset, Y \not\subseteq I$.

(i) If $|\langle \emptyset, \bar{I} \rangle| \leq |\mathcal{F}| / 2$ then there exists $[x] \subseteq I$ such that...

(ii) If $|\langle \emptyset, \bar{I} \rangle| \geq |\langle I, U \rangle|$ then there exists $[x] \subseteq \bar{I}$ such that...

$|\{Y \in \mathcal{F} : [x] \subseteq Y\}| \geq |\{Y \in \mathcal{F} : [x] \not\subseteq Y\}|$.

Proof: We assume that $Y \in \mathcal{F}$ implies $|Y| \geq 2$. Readily, $|\langle I, U \rangle| + |\mathcal{F} \sim \langle I, U \rangle| = |\mathcal{F}|$, and since $\langle \emptyset, \bar{I} \rangle \subseteq \mathcal{F} \sim \langle I, U \rangle$, $|\langle \emptyset, \bar{I} \rangle| \leq |\mathcal{F} \sim \langle I, U \rangle|$.

Let $I = \{x_1 x_2 \dots x_t\}$. We will run a finite induction on $x_i \in I$.

Initially, let $i = 1$ and let $Y \in \mathcal{F} \sim \langle I, U \rangle$.

(i) If $[x_1] \not\subseteq Y$ then $Y \subset \bar{I}$, because if $Y \in \langle I, U \rangle$, then $[x_1] \subset Y$. So, we have $|\{Y \in \mathcal{F} \sim \langle I, U \rangle : [x_1] \not\subseteq Y\}| = |\langle \emptyset, \bar{I} \rangle| = |\{Y \in \mathcal{F} : [x_1] \not\subseteq Y\}|$. Consequently, if $|\langle \emptyset, \bar{I} \rangle| \leq |\mathcal{F}| / 2 = (|\langle I, U \rangle| + |\mathcal{F} \sim \langle I, U \rangle|) / 2$ then $[x_1] \subseteq I$ is abundant.

(ii) Since $\langle \emptyset, \bar{I} \rangle$ is itself a U-C family we can assume $\exists x' \in \bar{I}$ where $[x']$ is an abundant set in $\langle \emptyset, \bar{I} \rangle$.

If $Y \in \mathcal{F} \sim \langle I, U \rangle$ and $Y \cap I \neq \emptyset$ then $Y = Y' \cup [x_1]$ for some $Y' \in \langle \emptyset, \bar{I} \rangle$. Thus $[x']$ must be in at least $(|\mathcal{F} \sim \langle I, U \rangle| + |\langle \emptyset, \bar{I} \rangle|) / 2$ sets of $\mathcal{F}$. So $[x']$ is abundant if $(|\mathcal{F} \sim \langle I, U \rangle| + |\langle \emptyset, \bar{I} \rangle|) / 2 \geq (|\mathcal{F} \sim \langle I, U \rangle| + |\langle I, U \rangle|) / 2 = |\mathcal{F}| / 2$ or equivalently, if $|\langle \emptyset, \bar{I} \rangle| \geq |\langle I, U \rangle|$.

(Figure 7 illustrates a U-C family with the properties of this initial configuration (b) where $Y \subseteq \{1235\}$ and $[x_1] = [5] \subset Y$. Increasing the number of sets in $\langle I, U \rangle = \langle [56], U \rangle$ would eventually yield condition (a).)

Thus both (a) and (b) are true when $i = 1$; we inductively assume they are true when $i = s < t$. Let $\mathcal{F}$ be any U-C family satisfying the condition that if $Y \cap I \neq \emptyset$ then $Y \cap I \subseteq \{x_1 x_2 \dots x_t\}$. We construct a smaller U-C family $\mathcal{F}' \subset \mathcal{F}$ as follows. For all Y in either $\langle I, U \rangle$ or $\langle \emptyset, \bar{I} \rangle$, let $Y = Y' \in \mathcal{F}'$. For $Y \in \mathcal{F} \sim \langle I, U \rangle$ (excluding $\langle \emptyset, \bar{I} \rangle$), if $[x_t] \not\subseteq Y$, again let $Y = Y' \in \mathcal{F}'$, but if $[x_t] \subset Y$ and $Y \setminus [x_t] \notin \mathcal{F} \sim \langle I, U \rangle$ (that is $Y \setminus [x_t]$ is not already a set of Y') then $Y \setminus [x_t] = Y' \in \mathcal{F}'$, otherwise $Y \notin \mathcal{F}'$.

Since $Y \in \mathcal{F}$ is deleted only if $Y \setminus [x_t] \in \mathcal{F} \sim \langle I, U \rangle$, Y is irreducible, and by Lemma 1.1, $\mathcal{F}'$ so created is U-C and $\mathcal{F}'$ satisfies the conditions of the proposition. Now consider $\mathcal{F}$ as an extension of $\mathcal{F}'$. $|\mathcal{F}'| \leq |\mathcal{F}|$ with equality if no irreducible sets of the form $Y \cup [x_t]$ exist in $\mathcal{F}$. Thus if $|\langle \emptyset, \bar{I} \rangle| \leq |\mathcal{F}'| / 2$ in $\mathcal{F}'$, then since $\langle \emptyset, \bar{I} \rangle$ is unchanged in $\mathcal{F}'$, $|\langle \emptyset, \bar{I} \rangle| \leq |\mathcal{F}| / 2$ in $\mathcal{F}$ and (a) follows. And since no sets of $\langle \emptyset, \bar{I} \rangle$ or $\langle I, U \rangle \in \mathcal{F}$ are deleted, if $|\langle \emptyset, \bar{I} \rangle| \geq |\langle I, U \rangle|$ in $\mathcal{F}'$ then $|\langle \emptyset, \bar{I} \rangle| \geq |\langle I, U \rangle|$ in $\mathcal{F}$, so conclusion (b) follows. $\square$

All possible configurations, when $\cap_j \{Z_j \in \mathcal{Z}\} = I \neq \emptyset$, have been accounted for. But this final proposition also demonstrates that whenever $|\langle I, U \rangle| \leq |\langle \emptyset, \bar{I} \rangle| \leq |\mathcal{F}| / 2$ we

are assured that abundant elements $[x_i]$ can be found in *both* I and $\bar{I}$.

5. EPILOGUE

Since either $\cap_j \{Z_j \in Z\} = \emptyset$, or not; taken together, Propositions 3.2, 4.1, 4.2, 4.3 and 4.4 resolve Frankl's U-C Conjecture. It is true — as almost everyone has thought it was. The other important result of this paper is uncovering the bifurcated nature of U-C families. They fall into two distinct structural classes, Class I, where $I = \cap_j \{Z_j \in Z\} \emptyset$, and Class II, where $I = \cap_j \{Z_j \in Z\} \neq \emptyset$. Because of this no single proof technique could work for both. In Class II, the atom set $\mathcal{A}$ tends to be small, so one looks for some abundant element $[x]$ in $\mathcal{A}$, frequently $[x] \subseteq I$. In Class I, $\mathcal{A}$ can be rather large, while $|Z|$ is limited by n , so one searches for an abundant element in $\bar{Z}$.

It appears that the relatively novel concepts of co-atoms, intervals and specifically the roles of $\langle \emptyset, \bar{I} \rangle$ and $\langle I, U \rangle$ will be useful tools in further investigations of these two classes of union-closed set families. And since the complement of union-closed families are intersection-closed, these discoveries are applicable to finite closure systems [3] as well.

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