

EXACT SOLUTIONS OF NONLINEAR COMPLEX PARTIAL DIFFERENTIAL EQUATIONS VIA TWO-DIMENSIONAL DIFFERENTIAL TRANSFORM METHOD

AMAL KHALAF HAYDAR¹, RADHI A. ZABOON², SHATHA S. ALHILY³

Abstract: This article introduces some results about two-dimensional differential transform for nonlinear complex functions and their derivatives which are needed in this paper to find exact solutions for nonlinear complex partial differential equations using two-dimensional differential transform method.

1. INTRODUCTION

There are many nonlinear phenomena that are important in applied science and engineering, including fluid motion, plasma therapy, rigid mechanics, and many others. This shows how partial differential equations control the models' potential in applications. Determining exact or approximate solutions to these forms of equations has received a lot of focus in recent studies. Understanding all analytical and numerical methods for solving nonlinear partial differential equations is therefore becoming increasingly substantial.

There are many methods that used for solving linear or nonlinear partial differential equations such as the Adomian decomposition method [1, 2, 17-20, 28], the variational iteration method [33, 34], the homotopy perturbation method [29], the reduced differential transform method [21, 22, 24, 31], the double integral transform known as DLST [3, 4, 12]. In addition, some researchers have merged an integral transform with other techniques in order to solve many nonlinear differential equations [5, 8, 11, 15, 16, 23, 26].

Two-dimensional transform method, which Chen and Ho proposed in 1999, is a useful tool for solving both linear and nonlinear partial differential equations [25], and it was widely used by several researchers such as, [6, 7, 10, 14, 30]. In 2021, Duz used the Elzaki transform method to solve an n th order complex partial differential equation with constant coefficients [9]. Also, in order to determine whether the solutions to some difficult second order nonlinear differential equations are topological spheres, McKay studied the relationship between the topological aspect and differential equations in 2006 [27]. Fang and Sun introduced the stability concept and the comparison principle of

¹ Department of Mathematics, Faculty of Education for Girls, University of Kufa, Najaf, Iraq.
Email: amalkh.hayder@uokufa.edu.iq

² Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq.
Email: r.zaboon@uomustansiriyah.edu.iq

³ Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq.
Email: shathamaths@uomustansiriyah.edu.iq

complex-valued autonomous systems [13]. Sun et al. have found sufficient conditions for nonlinear ordinary differential equation solutions to exist in some spaces [32]. In the present study, we are interested in solving nonlinear complex partial differential equations analytically using the two-dimensional differential transform method.

This paper is organized as follows, Section 2 consists of the basic definitions and theorems. Section 3 includes results about two-dimensional differential transform of a product of complex functions and derivatives. Section 4 uses the two-dimensional differential transform of a product of complex functions to find exact solutions to some nonlinear complex partial differential equations. Section 5 is conclusions.

2. BASIC DEFINITIONS AND CONCEPTS

In this section, we state the basic definitions and theorems that we needed in this work.

Definition 2.1. [25] Two-dimensional differential transform of a function $f(x, y)$ is defined as follows:

$$F(p, q) = \frac{1}{p! q!} \left. \frac{\partial^{p+q} f(x, y)}{\partial x^p \partial y^q} \right|_{x=0, y=0} \quad (2.1)$$

where $f(x, y)$ is the original function and $F(p, q)$ is the transformed function.

Definition 2.2. [25] Differential inverse transform of $F(p, q)$ is defined as follows:

$$f(x, y) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} F(p, q) x^p y^q \quad (2.2)$$

In fact from equations (2.1) and (2.2), we obtain

$$f(x, y) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{p! q!} \left. \frac{\partial^{p+q} f(x, y)}{\partial x^p \partial y^q} \right|_{x=0, y=0} x^p y^q$$

The above equation implies that the concept of two-dimensional differential transform is derived from the two-dimensional Taylor series expansion.

Theorem 2.1. [25] If $f(x, y) = u(x, y) \pm v(x, y)$ then $F(p, q) = U(p, q) \pm V(p, q)$.

Theorem 2.2. [25] If $f(x, y) = \lambda u(x, y)$ then $F(p, q) = \lambda U(p, q)$, where λ is a constant.

Theorem 2.3. [25] If $f(x, y) = \frac{\partial^{r+s} u(x, y)}{\partial x^r \partial y^s}$

then

$$F(p, q) = (p+1)(p+2) \cdots (p+r)(q+1)(q+2) \cdots (q+s) U(p+r, q+s).$$

Theorem 2.4. [25] If $f(x, y) = u(x, y)v(x, y)$ then

$$F(p, q) = \sum_{r=0}^p \sum_{s=0}^q U(r, q-s) V(p-r, s).$$

Theorem 2.5. [25] If $f(x, y) = x^m y^n$ then $F(p, q) = \delta(p-m, q-n)$.

Theorem 2.6. [9] Let $w = w(z)$ be a complex valued function with complex variables. Then

$$\begin{aligned} 1 - \frac{\partial^n w}{\partial z^n} &= \frac{1}{2^n} \sum_{k=0}^n (-i)^k \binom{n}{k} \frac{\partial^n w}{\partial x^{n-k} \partial y^k} \\ 2 - \frac{\partial^{n+r} w}{\partial z^n \partial \bar{z}^r} &= \frac{1}{2^{n+r}} \sum_{h=0}^n \sum_{k=0}^r (-i)^h i^k \binom{n}{h} \binom{r}{k} \frac{\partial^{n+r} w}{\partial x^{n+r-(h+k)} \partial y^{h+k}} \\ 3 - \frac{\partial^n w}{\partial \bar{z}^n} &= \frac{1}{2^n} \sum_{k=0}^n i^k \binom{n}{k} \frac{\partial^n w}{\partial x^{n-k} \partial y^k} \end{aligned}$$

3. TWO-DIMENSIONAL DIFFERENTIAL TRANSFORM FOR NONLINEAR COMPLEX FUNCTIONS

In this section, we present certain results regarding two-dimensional differential

transform of a product of two complex functions of the form $w(z, \bar{z}) = u(x, y) + iv(x, y)$, which is required in this paper for the solution of complex partial differential equations that are nonlinear.

Theorem 3.1. Let $w_1(z, \bar{z}) = u_1(x, y) + iv_1(x, y)$, $w_2(z, \bar{z}) = u_2(x, y) + iv_2(x, y)$. If $f(z, \bar{z}) = w_1(z, \bar{z})w_2(z, \bar{z})$, then $F(p, q) = \sum_{r=0}^p \sum_{s=0}^q W_1(r, q-s)W_2(p-r, s)$, where $W_i(p, q)$ is two-dimensional differential transform of $w_i(z, \bar{z})$, $i = 1, 2$.

Proof. We have:

$$\begin{aligned} f(z, \bar{z}) &= w_1(z, \bar{z})w_2(z, \bar{z}) \\ &= (u_1(x, y) + iv_1(x, y))(u_2(x, y) + iv_2(x, y)) \\ &= u_1(x, y)u_2(x, y) - v_1(x, y)v_2(x, y) + iu_1(x, y)v_2(x, y) + iv_1(x, y)u_2(x, y). \end{aligned}$$

By Theorem 2.4, we get:

$$F(p, q) = \sum_{r=0}^p \sum_{s=0}^q U_1(r, q-s)U_2(p-r, s) - \sum_{r=0}^p \sum_{s=0}^q V_1(r, q-s)V_2(p-r, s) + i \sum_{r=0}^p \sum_{s=0}^q U_1(r, q-s)V_2(p-r, s) + i \sum_{r=0}^p \sum_{s=0}^q V_1(r, q-s)U_2(p-r, s).$$

Then

$$\begin{aligned} F(p, q) &= \sum_{r=0}^p \sum_{s=0}^q [U_1(r, q-s)U_2(p-r, s) - V_1(r, q-s)V_2(p-r, s) + iU_1(r, q-s)V_2(p-r, s) + iV_1(r, q-s)U_2(p-r, s)] \\ &= \sum_{r=0}^p \sum_{s=0}^q [U_1(r, q-s) + iV_1(r, q-s)][U_2(p-r, s) + iV_2(p-r, s)] = \sum_{r=0}^p \sum_{s=0}^q W_1(r, q-s)W_2(p-r, s) \end{aligned}$$

Remark 3.1. In Theorem 3.1, $F(p, q)$ can be written also as:

$$1. F(p, q) = \sum_{r=0}^p \sum_{s=0}^q W_1(p-r, s)W_2(r, q-s)$$

or

$$2. F(p, q) = \sum_{r=0}^p \sum_{s=0}^q W_1(r, s)W_2(p-r, q-s).$$

The proofs are similar to proof of Theorem 3.1.

Concluding Remarks 3.1. From Theorem 3.1, we can get:

1. If $f(z, \bar{z}) = w^2(z, \bar{z})$, then $F(p, q) = \sum_{r=0}^p \sum_{s=0}^q W(r, q-s)W(p-r, s)$.

2. If $f(z, \bar{z}) = (\frac{\partial w(z, \bar{z})}{\partial z})^2$, then

$$\begin{aligned} F(p, q) &= \frac{1}{4} \sum_{r=0}^p \sum_{s=0}^q [(r+1)(p-r+1)W(r+1, q-s)W(p-r+1, s) - i(r+1)(s+1)W(r+1, q-s)W(p-r, s+1) \\ &\quad - i(q-s+1)(p-r+1)W(r, q-s+1)W(p-r+1, s) - (q-s+1)(s+1)W(r, q-s+1)W(p-r, s+1)]. \end{aligned}$$

3. If $f(z, \bar{z}) = (\frac{\partial w(z, \bar{z})}{\partial \bar{z}})^2$ then

$$\begin{aligned} F(p, q) &= \frac{1}{4} \sum_{r=0}^p \sum_{s=0}^q [(r+1)(p-r+1)W(r+1, q-s)W(p-r+1, s) + i(r+1)(s+1)W(r+1, q-s)W(p-r, s+1) \\ &\quad - i(q-s+1)(p-r+1)W(r, q-s+1)W(p-r+1, s) - (q-s+1)(s+1)W(r, q-s+1)W(p-r, s+1)]. \end{aligned}$$

4. If $f(z, \bar{z}) = w(z, \bar{z})w_z(z, \bar{z})$ then

$$F(p, q) = \frac{1}{2} \sum_{r=0}^p \sum_{s=0}^q [(p-r+1)W(p-r+1, s)W(r, q-s) + i(s+1)W(p-r, s+1)W(r, q-s)].$$

4. SOLVING NONLINEAR COMPLEX PARTIAL DIFFERENTIAL EQUATIONS BY TWO-DIMENSIONAL DIFFERENTIAL TRANSFORM METHOD

In this section, we shall use the results in the previous section to obtain analytical solutions of two complex nonlinear partial differential equations via the two-dimensional differential transform method.

Example 4.1. Consider the nonlinear complex IVP:

$$\frac{\partial^2 w}{\partial z^2} + \left(\frac{\partial w}{\partial z}\right)^2 - 4\bar{z}^2 \frac{\partial w}{\partial \bar{z}} = 2\bar{z}, \quad (4.1)$$

$$w(x, 0) = x^3 \quad (4.2)$$

$$\frac{\partial w}{\partial y}(x, 0) = ix^2 \quad (4.3)$$

Taking two-dimensional differential transform to both sides of equation (4.1) and using Theorem 2.6, Concluding Remarks 3.1 and Theorem 3.1, we obtain:

$$\begin{aligned} W(p, q+2) = & \frac{1}{(q+1)(q+2)} \left[-8 \frac{(-i)^q (p+q)!}{p! q!} \delta(p+q-1) + (p+1)(p+2)W(p+2, q) \right. \\ & - 2i(p+1)(q+1)W(p+1, q+1) + \sum_{r=0}^p \sum_{s=0}^q \{ (r+1)(p-r+1)W(r \\ & + 1, q-s)W(p-r+1, s) - i(r+1)(s+1)W(r+1, q-s)W(p-r, s \\ & + 1) - i(q-s+1)(p-r+1)W(r, q-s+1)W(p-r+1, s) - (q-s \\ & + 1)(s+1)W(r, q-s+1)W(p-r, s+1) \} \\ & \left. - 8 \sum_{r=0}^p \sum_{s=0}^q \frac{(-i)^{q-s} (r+q-s)!}{r! (q-s)!} \delta(r+q-s-2) \{ (p-r+1)W(p-r \right. \\ & \left. + 1, s) + i(s+1)W(p-r, s+1) \} \right], p, q = 0, 1, \end{aligned} \quad (4.4)$$

where

$$\delta(r+q-s-2) = \begin{cases} 1, & \text{if } r+q-s=2 \\ 0, & \text{if } r+q-s \neq 2 \end{cases}, \quad \delta(p+q-1) = \begin{cases} 1, & \text{if } p+q=1 \\ 0, & \text{if } p+q \neq 1 \end{cases}.$$

By the conditions (4.2) and (4.3), we can get:

$$W(3, 0) = 1 \text{ and } W(p, 0) = 0, \text{ for } p \neq 3, \quad (4.5)$$

and

$$W(2, 1) = i \text{ and } W(p, 1) = 0 \text{ for } p \neq 2, \quad (4.6)$$

respectively.

If we put $q = 0$ in equation (4.4), we get:

$$\begin{aligned} W(p, 2) = & \frac{1}{2} [-8\delta(p-1) + (p+1)(p+2)W(p+2, 0) - 2i(p+1)W(p+1, 1) + \sum_{r=0}^p \{ (r+1)(p-r \\ & + 1)W(r+1, 0)W(p-r+1, 0) - i(r+1)W(r+1, 0)W(p-r, 1) - i(p-r \\ & + 1)W(r, 1)W(p-r+1, 0) - W(r, 1)W(p-r, 1) \} \\ & - 8 \sum_{r=0}^p \delta(r-2) \{ (p-r+1)W(p-r+1, 0) + iW(p-r, 1) \}], p, q = 0, 1, 2, \dots \end{aligned}$$

which gives:

$$W(1, 2) = 1 \text{ and } W(p, 2) = 0 \text{ for } p \neq 1. \quad (4.7)$$

If we put $q = 1$ in equation (4.4), we get:

$$\begin{aligned} W(p, 3) = & \frac{1}{6} [8i(p+1)\delta(p) + (p+1)(p+2)W(p+2, 1) - 4i(p+1)W(p+1, 1) + \sum_{r=0}^p \sum_{s=0}^1 \{ (r+1)(p-r \\ & + 1)W(r+1, 1-s)W(p-r+1, s) - i(r+1)(s+1)W(r+1, 1-s)W(p-r, s+1) - i(2-s)(p-r+1) \\ & + 1)W(r, 2-s)W(p-r+1, s) - (2-s)(s+1)W(r, 2-s)W(p-r, s+1) \} - 8 \sum_{r=0}^p \sum_{s=0}^1 \frac{(-i)^{1-s} (r+1-s)!}{r! (1-s)!} \delta(r- \\ & 1-s) \{ (p-r+1)W(p-r+1, s) + i(s+1)W(p-r, s+1) \}], p = 0, 1, 2 \end{aligned}$$

which gives:

$$W(0, 3) = i \text{ and } W(p, 3) = 0 \text{ for } p \neq 0. \quad (4.8)$$

Similarly, if we put $q \geq 2$ in (4.4), we can get:

$$W(p, q) = 0 \text{ for } q \geq 4, p = 0, 1, 2, \quad (4.9)$$

Therefore, using Definition 2.2 and values of $W(p, q)$ in equations (4.5) -(4.9), the exact solution of the complex IVP (4.1) -(4.3) is:

$$\begin{aligned} w(z, \bar{z}) = & \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} W(p, q) x^p y^q = W(0, 0) + W(1, 0)x + W(2, 0)x^2 + W(3, 0)x^3 + W(4, 0)x^4 + \dots + [W(0, 1) + W(1, 1)x \\ & + W(2, 1)x^2 + W(3, 1)x^3 + W(4, 1)x^4 + \dots]y + [W(0, 2) + W(1, 2)x + W(2, 2)x^2 + W(3, 2)x^3 \\ & + W(4, 2)x^4 + \dots]y^2 + [W(0, 3) + W(1, 3)x + W(2, 3)x^2 + W(3, 3)x^3 + W(4, 3)x^4 + \dots]y^3 : \\ & = x^3 + ixyx^2 + y^2x + iy^3 = z^2\bar{z}. \end{aligned}$$

Example 4.2. Consider the nonlinear complex IVP:

$$\frac{\partial^3 w}{\partial z^3} + w \frac{\partial w}{\partial \bar{z}} - 8w = \bar{z}e^{4z}, \quad (4.10)$$

$$w(x, 0) = xe^{2x}, \quad (4.11)$$

$$\frac{\partial w}{\partial y}(x, 0) = 2ixe^{2x} - ie^{2x}, \quad (4.12)$$

$$\frac{\partial^2 w}{\partial y^2}(x, 0) = -4xe^{2x} + 4e^{2x}. \quad (4.13)$$

Taking two-dimensional differential transform to both sides of equation (4.10) and using Theorem 2.6, Concluding Remarks 3.1 and Theorem 3.1, we obtain:

$$\begin{aligned} W(p, q+3) = & \frac{-i}{(q+1)(q+2)(q+3)} \left[2 \frac{i^q 4^{p+q} (p-q)}{p! q!} \right. \\ & - (p+1)(p+2)(p+3)W(p+3, q) + 3i(p+1)(p+2)(q+1)W(p+2, q) \\ & + 1) + 3(p+1)(q+1)(q+2)W(p+1, q+2) + 64W(p, q) \\ & - 4 \sum_{r=0}^p \sum_{s=0}^q \{ (p-r+1)W(r, q-s)W(p-r+1, s) + i(s+1)W(r, q-s)W(p-r, s+1) \} \}, \quad p, q = 0, 1 \end{aligned} \quad (4.14)$$

By the initial conditions (4.11), (4.12) and (4.13), we get:

$$W(p, 0) = \frac{2^{p-1}p}{p!}, \quad p = 0, 1, 2 \quad (4.15)$$

$$W(p, 1) = \frac{i2^p(p-1)}{p!}, \quad p = 0, 1, 2, \quad (4.16)$$

and

$$W(p, 2) = \frac{2^p(-p+2)}{p!}, \quad p = 0, 1, 2, \quad (4.17)$$

respectively.

If we put $q = 0$ in equation (4.14), we get:

$$\begin{aligned} W(p, 3) = & \frac{-i}{6} [2 \cdot 4^p p \frac{1}{p!} - (p+1)(p+2)(p+3)W(p+3, 0) + 3i(p+1)(p+2)W(p+2, 1) + 6(p+1)W(p+1, 2) \\ & + 64W(p, 0) - 4 \sum_{r=0}^p \{ (p-r+1)W(r, 0)W(p-r+1, 0) + iW(r, 0)W(p-r, 1) \} \}, \quad p = 0, 1, \end{aligned}$$

Using equations (4.15), (4.16) and (4.17), the above equation gives:

$$W(p, 3) = \frac{-i2^p}{3p!} (2^p p + 2p - 6) + \frac{2i2^p}{3} \sum_{r=0}^p \frac{r}{r! (p-r)!} \quad (4.18)$$

which gives:

$$W(0, 3) = 2i, W(1, 3) = \frac{16i}{3!}, W(2, 3) = \frac{32i}{4!}, \quad (4.19)$$

If we put $q = 1$ in equation (4.14), we get:

$$\begin{aligned} W(p, 4) = & \frac{-i}{24} [8i \frac{4^p (p-1)}{p!} - (p+1)(p+2)(p+3)W(p+3, 1) + 6i(p+1)(p+2)W(p+2, 2) + 18(p+1)W(p+1, 3) \\ & + 64W(p, 1) - 4 \sum_{r=0}^p \sum_{s=0}^1 \{ (p-r+1)W(r, 1-s)W(p-r+1, s) + i(s+1)W(r, 1-s)W(p-r, s+1) \} \}, \quad p = 0, 1, \end{aligned}$$

Using (4.15) -(4.17) and (4.19) with achieving some calculations, the above equation becomes:

$$W(p, 4) = \frac{2^p}{3p!} (-2 \cdot 2^p p - 4 \cdot 2^p + 4p - 1) + \frac{2^p}{3} \sum_{r=0}^p \frac{1}{r! (p-r+1)!} (2r^2 + pr + p + 1), \quad p = 0, 1, 2, \quad (4.20)$$

which gives:

$$W(0, 4) = \frac{-8}{3!}, W(1, 4) = -\frac{48}{4!}, W(2, 4) = \frac{-4}{3}, \quad (4.21)$$

If we put $q = 2$ in equation (4.14), we get:

$$W(p, 5) = \frac{-i}{60} [-4^{p+2} \frac{(p-2)}{p!} - (p+1)(p+2)(p+3)W(p+3, 2) + 9i(p+1)(p+2)W(p+2, 3) + 36(p+1)W(p+1, 4) + 64W(p, 2) - 4 \sum_{r=0}^p \sum_{s=0}^2 \{(p-r+1)W(r, 2-s)W(p-r+1, s) + (p-r+1)W(r, 2-s)W(p-r, s+1)\}], p = 0, 1,$$

Using equations (4.15) - (4.17), (4.19) and (4.21) with achieving some calculations, the above equation becomes:

$$W(p, 5) = \frac{-2i2^p}{15p!} (6p^2 + 17p + 26 - 8 \cdot 2^p p - 20 \cdot 2^p) - \frac{i2^p}{30} \sum_{r=0}^p \frac{1}{r!(p+2-r)!} \{16p^2 r - 4p^2 - 8pr^2 + 68pr - 12p + 16r^3 - 40r^2 + 68r - 8 - 2^{p-r}(p^3 r - 3p^2 r^2 + 3p^2 r + 3pr^3 - 6pr^2 + 2pr - r^4 + 3r^3 - 2r^2)\} - \frac{i2^p}{15} \sum_{r=0}^p \sum_{m=0}^{p-r} \frac{mr}{r!m!(p-r-m)!}, p = 0, 1, \quad (4.22)$$

which gives:

$$W(0, 5) = \frac{-16i}{4!}, W(1, 5) = \frac{-16i}{15}, W(2, 5) = \frac{-4i}{5}, \quad (4.23)$$

By using Definition 2.1 and the values of $W(p, q)$ in equations (4.15) - (4.17), (4.19), (4.21) and (4.23), we get:

$$w(z, \bar{z}) = \sum_{p=0}^{\infty} \sum_{q=0}^2 W(p, q) x^p y^q + [W(0, 3) + W(1, 3)x + W(2, 3)x^2 + W(3, 3)x^3 + \dots] y^3 + [W(0, 4) + W(1, 4)x + W(2, 4)x^2 + W(3, 4)x^3 + \dots] y^4 + [W(0, 5) + W(1, 5)x + W(2, 5)x^2 + W(3, 5)x^3 + \dots] y^5 + \dots$$

$$= x + 2x^2 + \frac{4}{2!}x^3 + \frac{8}{3!}x^4 + \frac{16}{4!}x^5 + \dots - iy + i2x^2y + i\frac{16}{3!}x^3y + i\frac{48}{4!}x^4y + i\frac{128}{5!}x^5y + \dots + 2y^2 + \frac{4}{2!}xy^2 - \frac{32}{4!}x^3y^2 - \frac{32}{4!}x^4y^2 - \frac{96}{5!}x^5y^2 + \dots + 2iy^3 + i\frac{16}{3!}xy^3 + i\frac{32}{4!}x^2y^3 + \dots - \frac{8}{3!}y^4 - \frac{48}{4!}xy^4 - \frac{4}{3}x^2y^4 + \dots - \frac{16}{4!}iy^5 - \frac{16}{15}ixy^5 - \frac{4}{5}ix^2y^5 - \dots$$

Therefore, the exact solution of the complex IVP (4.10)-(4.13) is:

$$w(z, \bar{z}) = (x - iy) + (2x^2 + 2y^2) + \left(\frac{4}{2!}x^3 + i\frac{4}{2!}x^2y + \frac{4}{2!}xy^2 + \frac{4}{2!}iy^3\right) + \left(\frac{8}{3!}x^4 + i\frac{16}{3!}x^3y + i\frac{16}{3!}xy^3 - \frac{8}{3!}y^4\right) + \left(\frac{16}{4!}x^5 + i\frac{48}{4!}x^4y - \frac{32}{4!}x^3y^2 + i\frac{32}{4!}x^2y^3 - \frac{48}{4!}xy^4 - \frac{16}{4!}iy^5\right) + \dots$$

$$= x[1 + 2(x + iy)\frac{4}{2!}(x + iy)^2 + \frac{8}{3!}(x + iy)^3 + \frac{16}{4!}(x + iy)^4 + \dots] - iy[1 + 2(x + iy) + \frac{4}{2!}(x + iy)^2 + \frac{8}{3!}(x + iy)^3 + \frac{16}{4!}(x + iy)^4 + \dots]$$

$$= (x - iy)[1 + 2(x + iy) + \frac{4}{2!}(x + iy)^2 + \frac{8}{3!}(x + iy)^3 + \frac{16}{4!}(x + iy)^4 + \dots] = \bar{z}e^{2z}$$

5. CONCLUSIONS

Two-dimensional differential transform of a product two complex functions which we have founded in this article can be used for finding two-dimensional differential transform of a product for any finite number of complex functions by making a specific selection of the complex functions. Then, we can use this product to solve and to find exact solutions to a widely nonlinear complex partial differential equations.

ACKNOWLEDGEMENTS

The authors would like to thank Mustansiriyah University (www.uomustansiriyah.edu.iq) Baghdad – Iraq for its support in the present work.

REFERENCES

- [1] G. Adomian, *Nonlinear dissipative wave equations*, Applied mathematics letters **11** (1998), 125-126.
- [2] G. Adomian, *A review of the decomposition method and some recent results for nonlinear equations*, Mathematical and Computer Modelling **13** (1990), 17-43.
- [3] S. A. Ahmed, Applications of New Double Integral Transform (Laplace–Sumudu Transform) in Mathematical Physics, Abstract and Applied Analysis, Hindawi Limited, 2021, pp. 1-8.
- [4] S. A. Ahmed, T. M. Elzaki, M. Elbadri, M. Z. Mohamed, *Solution of partial differential equations by new double integral transform (Laplace - Sumudu transform)*, Ain Shams Engineering Journal **12** (2021), 4045-4049.
- [5] L. Ali, S. Islam, T. Gul, I. S. Amiri, *Solution of nonlinear problems by a new analytical technique using Daftardar-Gejji and Jafari polynomials*, Advances in Mechanical Engineering **11** (2019), 1687814019896962.
- [6] M. Bagheri, J. Manafianheris, *Differential Transform Method for Solving the Linear and Nonlinear Westervelt Equation*, Journal of Mathematical Extension **6** (2012), 81-91.
- [7] A. Borhanifar, R. Abazari, *Exact solutions for non-linear Schrödinger equations by differential transformation method*, Journal of Applied Mathematics and Computing **35** (2011), 37-51.
- [8] R. R. Dhunde, G. L. Waghmare, *Double Laplace Iterative Method for Solving Nonlinear Partial Differential Equations*, New Trends Math. Sci **7** (2019), 138-149.
- [9] M. Düz, *Solution of complex partial derivative equations with constant coefficients via Elzaki Transform method*, J. App. and Eng. Math **11** (2021), 228-236.
- [10] A. Ebaid, *On A New Differential Transformation Method For Solving Nonlinear Differential Equations*, Asian-European Journal of Mathematics **6** (2013), 1350057.
- [11] H. Eltayeb, *A note on double Laplace decomposition method and nonlinear partial differential equations*, New Trends in Mathematical Sciences **5** (2017), 156-164.
- [12] H. Eltayeb, A. Kiliçman, *A note on double Laplace transform and telegraphic equations*, Abstract and applied analysis, Hindawi, 2013.
- [13] T. Fang, J. Sun, *Stability analysis of complex-valued nonlinear differential system*, Journal of Applied Mathematics **2013** (2013), 1-7.
- [14] D. D. Ganji, R. Hasankhanigavabari, A. Bozorgi, *Applications of The Two-Dimensional Differential Transform And Least Square Method for Solving Nonlinear Wave Equations*, New Trends in Mathematical Sciences **2** (2014), 95-105.
- [15] A. E. Hamza, T. M. Elzaki, *Application of Homotopy perturbation and Sumudu transform method for solving Burgers equations*, American Journal of Theoretical and Applied Statistics **4** (2015), 480-483.
- [16] E. Hilal, T. M. Elzaki, *Solution of nonlinear partial differential equations by new Laplace variational iteration method*, Journal of Function Spaces **2014** (2014), 1-5.
- [17] D. Kaya, *An explicit and numerical solutions of some fifth-order KdV equation by decomposition method*, Applied Mathematics and Computation **144** (2003), 353-363.
- [18] D. Kaya, *A New Approach To Solve A Nonlinear Wave Equation*, Bulletin of

- the Malaysian Mathematical Society. Second Series **21** (1998), 95-100.
- [19] D. Kaya, M. Aassila, *An application for a generalized KdV equation by the decomposition method*, Physics Letters A **299** (2002), 201-206.
 - [20] D. Kaya, M. Inc, *On the solution of the nonlinear wave equation by the decomposition method*, Bulletin of the Malaysian Mathematical Society. Second Series **22** (1999), 151-155.
 - [21] Y. Keskin, G. Oturanc, *Reduced Differential Transform Method For Solving Linear And Nonlinear Wave Equations*, Iranian Journal of Science & Technology **34** (2010), 113-122.
 - [22] Y. Keskin, G. Oturanç, *Reduced Differential Transform Method for Generalized KdV Equations*, Mathematical and Computational Applications, 2010, pp. 382-393.
 - [23] Y. Khan, Q. Wu, *Homotopy perturbation transform method for nonlinear equations using He's polynomials*, Computers & Mathematics with Applications **61** (2011), 1963-1967.
 - [24] M. M. Khater, *Diverse bistable dark novel explicit wave solutions of cubic-quintic nonlinear Helmholtz model*, Modern Physics Letters B **35** (2021), 2150441.
 - [25] C. o. Kuang Chen, S. Huei Ho, *Solving partial differential equations by two-dimensional differential transform method*, Applied Mathematics and Computation **106** (1999), 171-179.
 - [26] H. Kumar Mishra, A. K. Nagar, *He-Laplace method for linear and nonlinear partial differential equations*, Journal of Applied Mathematics **2012** (2012), 1-16.
 - [27] B. McKay, *Complex Nonlinear Ordinary Differential Equations And Geometry*, Journal of Physics: Conference Series, IOP Publishing, 2006, pp. 165.
 - [28] S. Pamuk, *Solution of the porous media equation by Adomian's decomposition method*, Physics Letters A **344** (2005), 184-188.
 - [29] M. Rafei, D. D. Ganji, *Explicit Solutions of Helmholtz Equation and Fifth-order KdV Equation using Homotopy Perturbation Method*, International Journal of Nonlinear Sciences and Numerical Simulation **7** (2006), 321-328.
 - [30] A. S. V. Ravi Kanth, K. Aruna, *Differential transform method for solving the linear and nonlinear Klein-Gordon equation*, Computer Physics Communications **180** (2009), 708-711.
 - [31] R. Z. Saadeh, B. F. Ghazal, *A New Approach on Transforms: Formable Integral Transform and Its Applications*, Axioms, 2021, pp. 332.
 - [32] Y. Sun, J. Long, G. Hu, *Nonlinear complex differential equations with solutions in some function spaces*, Complex Variables and Elliptic Equations **64** (2018), 300-314.
 - [33] A.-M. Wazwaz, *The variational iteration method: A powerful scheme for handling linear and nonlinear diffusion equations*, Computers & Mathematics with Applications **54** (2007), 933-939.
 - [34] A.-M. Wazwaz, *The variational iteration method: A reliable analytic tool for solving linear and nonlinear wave equations*, Computers & Mathematics with Applications **54** (2007), 926-932.