

ANALYTICAL SOLUTION LAMBERT W FUNCTION WITH TARIG TRANSFORMATION FOR DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this study, a method for solving linear scalar delay differential equations analytically utilizing the Lambert W function and Tarig transformation has been proposed. Examples of the approach's validation have been provided, and they emphasize how the solution method can be compared to systems of ordinary differential equations.

INTRODUCTION

The Lambert W function, $W(z)$, was first developed in the 1700s by Lambert and Euler [1, 13]. Studied authors Yi and Nelson in 2006 method using the Lambert W function for the analytical free and forced solution of systems of delay differential equations with a single delay. provided a solution in the Laplace domain using the analytical solution, provide examples of validation and underline how the solution process is similar to systems of ordinary differential equations [15]. In 2010 presented the book which includes compilation of recent studies on the creation of an analytical method for Lambert W function-based delay differential equation solutions. It also covers analysis and control techniques based on the solutions, as well as how to apply them to mechanical and biological systems [13]. How to find the roots of the delayed argument linear matrix differential equation, which corresponds to the matrix transcendental characteristic equation using Lambert W function, was presented by authors Ivanovien and Rimas in 2011 [6]. In 2014 use Jadlovská the characteristics Lambert W function, the fundamental ideas and findings concerning the oscillatory nature of solutions as well as stability analysis are revisited and contrasted with earlier findings [7]. In 2017 Pušenjak applied Lambert functions as an analytical technique to address the issue of production system control with delay. The results show that production delay might, depending on its extent, generate an oscillatory or aperiodic time response of actual inventory [10]. Similar to the complex logarithm, the Lambert W function, indicated by $W(z)$, can be defined as the function's multivalued inverse $z = \psi(w) = we^w$, where it is assumed that the variables z and w are complex, fulfilling the 'defining equation'

$$z = W(z)e^{W(z)} \quad (1)$$

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Consequently, the aforementioned special function, also referred to as a product log function, which unequivocally demonstrates a natural logarithmic expansion, is an illustration of a multivalued function $\psi(z)$, which, in contrast to single-valued functions, have two or more different values for each value of z . Infinite branches with $k \in \mathbb{Z}$ as W'_k 's index serve to illustrate the multivaluedness.

The multivaluedness is represented by infinitely many branches indexed by $k \in \mathbb{Z}$ as W'_k 's, which restrict the original domain to smaller subsets of the complex plane, on which our function is already single-valued and continuous [7]. The Lambert W function's branches are identified by the notation $W_k(z)$, where k takes the values $k = -\infty, \dots, -1, 0, 1, \dots, \infty$. The Lambert W function's principal branch, also known as the fundamental branch, has the index $k = 0$ and is represented by the notation $W_0(z)$ [13]. In the complex plane, the branches of $W_k(z)$ are precisely specified by $Re[W_k(z)] \rightarrow Im[W_k(z)]$, however, the real and imaginary components of $W_k(z)$, $k = 0, -1$ depending on z (i.e. $z \rightarrow Re[W_k(z)]$ and $z \rightarrow Im[W_k(z)]$), are the main focus of our concern [7].

LAMBERT W FUNCTION: A BRIEF OVERVIEW

Consider the first-order scalar delay differential equations (DDEs):

$$\dot{x}(t) = ax(t) + a_d x(t - \tau) + bu(t) \quad (2)$$

where the known delay τ is constant and the parameters a, b and a_d are constants.

The initial condition $x(t = 0) = x_0$

and preshape function $x(t) = g(t)$ for $-\tau < t < 0$, definitely the Lambert The transcendental characteristic equation of Eq. (2) is solved using the W function and is expressed as follows:

$$(s - a)e^{s\tau} = -a_d \quad (3)$$

When we multiply both sides of Equation (3) by $e^{-a\tau}$, we get

$$(s - a)\tau e^{\tau(s-a)} = -a_d \tau e^{-a\tau} \quad (4)$$

It is clear from the way the Lambert W function is expressed in Eq. (1)

$$W(a_d \tau e^{-a\tau}) e^{W(a_d \tau e^{-a\tau})} = a_d \tau e^{-a\tau} \quad (5)$$

Comparing Eqs. (4) and (5)

$$(s - a)\tau = W(a_d \tau e^{-a\tau}) \quad (6)$$

Therefore, in Equation (3), the solution to the characteristic equation can expressed as follows using the Lambert W function:

$$s = \frac{1}{\tau} W(a_d \tau e^{-a\tau}) + a \quad (7)$$

The scalar DDE's infinite spectrum is found in Eq. (2) by using the infinite branches of the Lambert W function, it is explicitly supplied in terms of the parameters a, a_d and τ of the system. The characteristic Eq. (3) has the following roots for $k = 0, \pm 1, \pm 2, \pm 3, \dots, \pm \infty$

$$s_k = \frac{1}{\tau} W_k(a_d \tau e^{-a\tau}) + a \quad (8)$$

Furthermore, the s -plane's rightmost eigenvalue, which is the main factor in determining stability, which, as demonstrated by Shinozaki and Mori [11], for Eq. (2), can be accomplished by utilizing the principal branch of the Lambert W function, where $k = 0$ [12]. In the following explain the properties of eigenvalues of S_k When a, a_d are constant:

i. $Re(S_0) \geq Re(S_k)$, $k = \pm 1, \pm 2, \dots$, for any given a, a_d and T .

ii. $a_d \tau e^{-a\tau} > -e^{-1} \Leftrightarrow S_0 \in \mathbb{R}$, $a_d \tau e^{-a\tau} = -e^{-1} \Leftrightarrow S_0 = -\frac{1}{\tau} + a$, $a_d \tau e^{-a\tau} < -e^{-1} \Leftrightarrow S_0 \in \mathbb{C}$.

iii. $-\frac{1}{\tau} + a \leq Re(S_0) < |a| + |a_d|$.

iv. $0 \leq Im(S_0) < \frac{\pi}{\tau}$

TARIG TRANSFORMATION

We suggested a novel integral transformation technique, called Tarig transformation, in light of the several intriguing methods used to solve differential and integral equations that facilitate visualization. Following that, this approach is used to resolve differential equations [2]. The Tarig transformation method is particularly efficient for solving a integral equations and linear system of differential as well as differential and integral equations. Introducing Elzaki and Elzaki [2] Tarig transformation technique, which is based on the Fourier transform [3, 5]. The Tarig transformation technique is defined for functions of exponential order by the following integral equations:

$$T[f(x)] = F(u) = \frac{1}{u} \int_0^{\infty} f(x) e^{\frac{-x}{u^2}} dx, x \geq 0, u \neq 0$$

In this transformation, the variable u serves as the transformation variable. The Laplace transformation and this transformation are more closely related [3]. Tarig transformation may or may not occur because the requirements for its existence are that f be exponential order and piecewise continuous. All issues that the well-known and frequently employed Laplace transformation typically addresses can be treated by a trig transformation, without a doubt [4, 5].

Finding solutions to linear DDEs with constant coefficients can be done with the help of the Tarig transform. With the use of the Lambert W function, the Tarig transform method for linear DDE systems is developed. This approach enables the analytical solution of DDEs, clarifies the features of delayed systems, which is intended to make it possible to analyze a system of DDEs for pole, positioning, controllability, etc.

SOLUTION OF LINEAR SCALAR DDEs USING APPROACH LAMBERT W FUNCTION WITH TARIG TRANSFORMATION

Free Solution for Scalar DDE

First, in the case of a first-order scalar homogeneous DDE, $\dot{x}(t) + ax(t) + a_d x(t - \tau) = 0$, $t > 0$

$$x(t) = \phi(t), \quad t \in [-\tau, 0] \quad (9)$$

The free solution terms of the ODEs Lambert W function, W_k (Asi and Ulsoy, 2003),

$$x(t) = \sum_{k=-\infty}^{\infty} C_k^I e^{\left(\frac{1}{\tau} W_k(-a_d \tau e^{a\tau}) - a\right)t} \quad (10)$$

Where the C_k^I are determined from the preshape function $\phi(t)$, where $s_k = \frac{1}{\tau} W_k(-a_d \tau e^{a\tau}) - a$. [14]

Consider the scalar free DDE in Eq. (2). Since $x(t)$ typically has a non-zero preshape function, the following analytical approach to compute C_k^I based on the Tarig transform can be used to transform it:

$$\int_0^{\infty} e^{-\frac{t}{u}} x((t - \tau)u) dt = \int_0^{\tau} e^{-\frac{t}{u}} x((t - \tau)u) dt + \int_{\tau}^{\infty} e^{-\frac{t}{u}} x((t - \tau)u) dt$$

$$\text{For } \int_{\tau}^{\infty} e^{-\frac{t}{u}} x((t - \tau)u) dt$$

$$\text{Let } w = t - \tau \rightarrow dw = dt, \text{ when } \begin{cases} t = \infty \rightarrow w = \infty - \tau = \infty \\ t = \tau \rightarrow w = \tau - \tau = 0 \end{cases}$$

$$\text{then } \int_{t=\tau}^{\infty} e^{-\frac{t}{u}} x((t - \tau)u) dt = \int_{w=0}^{w=\infty} e^{-\frac{t(w+\tau)}{u}} x(wu) dw$$

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$$\int_{\tau}^{\infty} e^{-\frac{t}{u}} x((t - \tau)u) dt = e^{-\frac{\tau}{u}} \int_0^{\infty} e^{-\frac{t}{u}} x(tu) dt$$

$$\begin{aligned}
& \Rightarrow \int_0^\tau e^{-\frac{t}{u}} \phi(tu) dt + \int_0^\infty e^{-\frac{-(t+\tau)}{u}} x(tu) dt \\
& = \int_0^\infty e^{-\frac{t}{u}} \phi(tu) dt + e^{-\frac{\tau}{u}} \int_0^\infty e^{-\frac{t}{u}} x(tu) dt \\
& = \Phi(u) + e^{-\frac{\tau}{u}} X(u) \quad (11)
\end{aligned}$$

Since $\Phi(u)$ define on interval $[-\tau, 0]$ then equal to zero except this and subsequently

$$\int_0^\infty e^{-\frac{t}{u}} \phi(tu) dt = \int_0^\tau e^{-\frac{t}{u}} \phi(tu) dt$$

The transform of Eq. (9) using by Tarig Transform can thus be expressed as:

$$\frac{1}{u^2} X(u) - \frac{1}{u} x(0) + a_d e^{-\frac{\tau}{u}} X(u) + a_d \Phi(u) + a X(u) = \left(\frac{1}{u^2} + a_d e^{-\frac{\tau}{u}} + a \right) X(u) - \frac{1}{u} x(0) + a_d \Phi(u) = 0 \quad (12)$$

Hence

$$X(u) = \frac{\frac{1}{u} x(0) - a_d \Phi(u)}{\frac{1}{u^2} + a_d e^{-\frac{\tau}{u}} + a} \quad (13)$$

On the other hand, the Tarig Transform can be used to transform the solution to Eq. (10) obtained by employing the Lambert W function:

$$\begin{aligned}
X(u) &= \sum_{k=-\infty}^{\infty} \frac{u C_k}{1 - S_k u^2} \\
X(u) &= \frac{u C_{-2}}{1 - S_{-2} u^2} + \frac{u C_{-1}}{1 - S_{-1} u^2} + \frac{u C_0}{1 - S_0 u^2} + \frac{u C_1}{1 - S_1 u^2} + \frac{u C_2}{1 - S_2 u^2} + \quad (14)
\end{aligned}$$

Where $S_k = \frac{1}{\tau} W_k(-a_d \tau e^{a\tau}) - a$

A common denominator can be used to reduce the fractions.

$$X(u) = \sum_{k=-\infty}^{\infty} \frac{C_k n_k(u)}{d(u)} \quad (15)$$

By defining,

$$\begin{aligned}
d(u) &\equiv \prod_{k=-\infty}^{\infty} (1 - S_k u^2) = (1 - S_{-2} u^2)(1 - S_{-1} u^2)(1 - S_0 u^2)(1 - S_1 u^2)(1 - S_2 u^2) \\
n_k(u) &= \frac{d(u)}{(1 - S_k u^2)} = (1 - S_{k-2} u^2)(1 - S_{k-1} u^2)(1 - S_{k+1} u^2)(1 - S_{k+2} u^2) \quad (16)
\end{aligned}$$

Where the polynomials in the Tarig variable u are, $n_k(u)$ and $d(u)$. The two equations Eq. (13) and Eq. (15) can now be compared. By doing so, we get:

$$X(u) = \sum_{k=-\infty}^{\infty} \frac{C_k n_k(u)}{d(u)} = \frac{\frac{1}{u} x(0) - a_d \Phi(u)}{\frac{1}{u^2} + a_d e^{-\frac{\tau}{u}} + a} \quad (17)$$

In Eq. (17), comparing the denominators on both sides results in:

$$d(u) \equiv \prod_{k=-\infty}^{\infty} (1 - S_k u^2) = J(u) \left(\frac{1}{u^2} + a_d e^{-\frac{\tau}{u}} + a \right) \quad (18)$$

And

$$\sum_{k=-\infty}^{\infty} C_k \cdot n_k = J(u) \left\{ \frac{1}{u} x(0) - a_d \Phi(u) \right\} \quad (19)$$

Where $J(u)$ is an undetermined polynomial in u . Eq. (16) states that n_k has the fascinating and practical characteristic,

$$n_k(u = S_l) = 0 \quad \text{when } k \neq l \\ = (1 - S_{k-2} S_l^2)(1 - S_{k-1} S_l^2)(1 - S_{k+1} S_l^2)(1 - S_{k+2} S_l^2) \text{ when } k = l \quad (20)$$

This characteristic allows us to calculate C_k^l from Eq. (19) by changing S_k for u , which is

$$\begin{aligned}
& \vdots \\
C_0^l &= \frac{J(S_0) \left\{ \frac{1}{S_0} x(0) - a_d \Phi(S_0) \right\}}{n_0(S_0)} = \frac{J(S_0) \left\{ \frac{1}{S_0} x(0) - a_d \Phi(S_0) \right\}}{(1 - S_{-2} S_0^2)(1 - S_{-1} S_0^2)(1 - S_1 S_0^2)(1 - S_2 S_0^2)(1 - S_3 S_0^2)} \\
C_1^l &= \frac{J(S_1) \left\{ \frac{1}{S_1} x(0) - a_d \Phi(S_1) \right\}}{n_1(S_1)} = \frac{J(S_1) \left\{ \frac{1}{S_1} x(0) - a_d \Phi(S_1) \right\}}{\dots (1 - S_{-1} S_1^2)(1 - S_0 S_1^2)(1 - S_2 S_1^2)(1 - S_3 S_1^2)(1 - S_4 S_1^2)} \quad (21)
\end{aligned}$$

Using Eq. (18), $J(u)$ is computed as

$$\prod_{k=-\infty}^{\infty} (1 - S_k u^2) = J(u) \left(\frac{1}{u^2} + a_d e^{-\frac{\tau}{u}} + a \right) \Leftrightarrow J(u) = \frac{\prod_{k=-\infty}^{\infty} (1 - S_k u^2)}{\left(\frac{1}{u^2} + a_d e^{-\frac{\tau}{u}} + a \right)} \quad (22)$$

And using L'Hopital's rule,

$$J(u) = \lim_{u \rightarrow S_0} \frac{\prod_{k=-\infty}^{\infty} (1 - S_k u^2)}{\left(\frac{1}{u^2} + a_d e^{-\frac{\tau}{u}} + a\right)} = \lim_{u \rightarrow S_0} \frac{\frac{\partial}{\partial u} \prod_{k=-\infty}^{\infty} (1 - S_k u^2)}{\frac{\partial}{\partial u} \left(\frac{1}{u^2} + a_d e^{-\frac{\tau}{u}} + a\right)}$$

$$= \frac{(1-S_{-2}S_0^2)(1-S_{-1}S_0^2)(1-S_1S_0^2)(1-S_2S_0^2)(1-S_3S_0^2)}{-\frac{2}{u^3} + \frac{1}{u^2} a_d \tau e^{-\frac{\tau}{u}}} \quad (23)$$

If we enter the aforementioned result into equation (21), we get

$$C_k^I = \frac{\frac{1}{S_k} x(0) - a_d \Phi(S_k)}{-\frac{2}{S_k^3} + \frac{1}{S_k^2} a_d \tau e^{-\frac{\tau}{S_k}}} \quad (24)$$

Forced Solution for Scalar DDE

Consider non-homogenous equation

$$\begin{aligned} \dot{x}(t) + ax(t) + a_d x(t - \tau) &= bu(t), t > 0 \\ x(t) &= \emptyset(t), \quad t \in [-\tau, 0] \\ x(t) &= x_0 \quad t = 0 \end{aligned} \quad (25)$$

where continuous function called $u(t)$ represents external stimulation. Malek-Zavarei and Jamshidi's [8] authors offer the following as the forced solution to Equation (9):

$$x(t) = \int_0^t \sum_{k=-\infty}^{\infty} e^{S_k(t-\zeta)} C_k^N bu(\zeta) d\zeta \quad (26)$$

Consequently, the Eq. (2) total solution is

$$x(t) = \underbrace{\sum_{k=-\infty}^{\infty} e^{S_k t} C_k^I}_{free} + \underbrace{\int_0^t \sum_{k=-\infty}^{\infty} e^{S_k(t-\zeta)} C_k^N bu(\zeta) d\zeta}_{forced} \quad (27)$$

Eq. (27) demonstrates that the form of the total solution of DDEs using the Lambert W function is similar to that of ODEs. In contrast to C_k^N , the coefficients C_k^I are dependent on the beginning conditions and the preshape function [13, 14].

The following part includes the calculated C_k^N along with an analytical approach for computing C_k^I and C_k^N based on the Tarig transform: In light of the aforementioned finding, let's have a look at the scalar nonhomogenous case in Eq. (25), the Tarig transform of Eq. (25) is:

$$\frac{1}{u^2} X(u) - \frac{1}{u} x(0) + a_d e^{-\frac{\tau}{u}} X(u) + a_d \Phi(u) + aX(u) = \left(\frac{1}{u^2} + a_d e^{-\frac{\tau}{u}} + a\right) X(u) - \frac{1}{u} x(0) + a_d \Phi(u) = bU(u) \quad (28)$$

Consequently, the response's transform, $x(t)$, is

$$X(u) = \underbrace{\frac{\frac{1}{u} x(0) - a_d \Phi(u)}{\frac{1}{u^2} + a_d e^{-\frac{\tau}{u}} + a}}_{free} + \underbrace{\frac{bU(u)}{\frac{1}{u^2} + a_d e^{-\frac{\tau}{u}} + a}}_{forced} \quad (29)$$

According to the Tarig transformation's transform of the convolution property, the inverse of the forced Eq. (29) is:

$$\frac{bU(u)}{\frac{1}{u^2} + a_d e^{-\frac{\tau}{u}} + a} = \frac{1}{\frac{1}{u^2} + a_d e^{-\frac{\tau}{u}} + a} bU(u) \Leftrightarrow \int_0^t \sum_{k=-\infty}^{\infty} e^{S_k(t-\zeta)} C_k^N bu(\zeta) d\zeta \quad (30)$$

Where, C_k^N is also obtained in the same way for C_k^I (case homogenous) as:

$$C_k^N = \frac{1}{-\frac{2}{S_k^3} + \frac{1}{S_k^2} a_d \tau e^{-\frac{\tau}{S_k}}} \quad (31)$$

Note that whereas C_k^I depends on the initial condition, $x(0)$ and the pershape function $\emptyset(t)$, but C_k^N is not,

$$\frac{1}{-\frac{2}{S_k^3} + \frac{1}{S_k^2} a_d \tau e^{-\frac{\tau}{S_k}}} = \sum_{k=-\infty}^{\infty} e^{S_k t} C_k^N \quad (32)$$

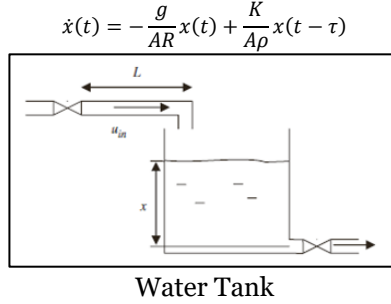
APPLICATION

A water tank and pipe of length L are depicted in Figure 1.1. For simulating the water tank, a linear model is taken into account.

$$\dot{x}(t) = \frac{1}{A\rho} \left(u_{in}(t - \tau) - \frac{\rho g x(t)}{R} \right)$$

where R is the hydrodynamic resistance of the output pipe, u_{in} is the input flow rate,

$x(t)$ is the water surface level, and A is the cross-sectional area, ρ is the water density, g is gravitational acceleration, τ is the delay, and t is time. The control's goal is to keep the water level within the tank. Assuming a proportional state feedback control, $u_{in}(t) = kx(t)$, a delay differential equation (DDE) can be used to describe the closed loop sys



Let

$$A = 10 \text{ m}^2, \rho = 1000 \text{ kgm}^{-3}, g = 10 \text{ ms}^{-2}, R = 10 \text{ m}^{-\frac{1}{2}} \text{ kg}^{-\frac{1}{2}}, \\ \tau = 1 \text{ s}, K = -1000.$$

Therefore, the closed-loop system is

$$\dot{x}(t) = -\frac{g}{AR}x(t) + \frac{K}{A\rho}x(t - \tau) = -0.1x(t) - 0.1x(t - 1)$$

We want to derive the analytical solution with initial state $x(0) = 10$,

$$g(t) \equiv 0 \text{ for } t \in [-\tau, 0] \text{ [9]}$$

Solution: from Eqs. (7)(24), (31) and by using Matlab software we can computing two coefficients S_k , C_k^I and C_k^N when $k = 0$

$$S_0 = \frac{1}{1}W_0(-0.1 \cdot 1 \cdot e^{-0.1}) + 0.1 = -0.2253$$

$$C_k^I = \frac{\frac{1}{S_0}x(0) - a_d\Phi(S_k)}{-\frac{2}{S_k^3} + \frac{1}{S_k^2}a_d\tau e^{-\frac{\tau}{S_k}}}$$

$$C_0^I = \frac{\frac{1}{S_0}x(0) - a_d\Phi(S_0)}{-\frac{2}{S_0^3} + \frac{1}{S_0^2}a_d\tau e^{-\frac{\tau}{S_0}}} = \frac{\frac{1}{S_0}x(0) - a_d e^{-\frac{\tau}{S_0}} \int_0^\tau e^{-\frac{\tau}{S_0}} g(t - \tau) dt}{-\frac{2}{S_0^3} + \frac{1}{S_0^2}a_d\tau e^{-\frac{\tau}{S_0}}}$$

$$= \frac{-\frac{1}{0.2253}(10) - 0.1e^{\frac{1}{0.2253}} \int_0^1 e^{\frac{1}{0.2253}} g(t - 1) dt}{-\frac{2}{(-0.2253)^3} + \frac{1}{(-0.2253)^2} 0.1e^{\frac{1}{0.2253}}} = -5.5260$$

$$C_0^N = \frac{1}{-\frac{2}{S_0^3} + \frac{1}{S_0^2}a_d\tau e^{-\frac{\tau}{S_0}}} = \frac{1}{8.0333} = 0.1245$$

The next table contains the results for some other branches:

Branch K	2	1	0	-1	-2
S_k	-4.9861 +13.7968i	-4.4439 + 7.3183i	- 0.2253	-3.53722	- 4.4439- 7.3183i
C_k^I	1.0923e+03 - 9.1112e+01i	3.6530e+02 + 2.6502e+02i	-5.5260	-81.7413	3.6530e+02 - 2.6502e+02i
C_k^N	-4.1891e+02 + 1.5524e+03i	-3.5628e+02 + 1.4957e+02i	0.1245	28.9140	-3.5628e+02 - 1.4957e+02i

Consequently, the following equation can be used to compute the system's total

solution as Eq. (27), and we find the eigenvalues from above table satisfies properties of eigenvalue of S_k .

SUMMARY

We presented a Lambert W function-based method for the study analysis and control of time-delay systems and the obtained analytical solutions were derived systematically for delayed differential equations. The eigenvalues of the principal branch of the Lambert W function were used to determine stability.

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