Math. Appl. 15(1) 2026, 152-180

Degree Sequences and Secure Domination Variants in Connected Graphs of Order at Most Five

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Received: October 2025, Accepted: December 2025, Published: Jan 2026.

Abstract

Secure domination models a movable guard system: every unoccupied vertex must have a neighboring guard that can move to it without destroying the required protection. This paper gives a complete, graph-by-graph classification of the secure domination number, secure total domination number, and secure connected domination number for every connected non-isomorphic simple graph of order from two through five. The corpus contains exactly thirty graphs: one of order two, two of order three, six of order four, and twenty-one of order five. For each graph, the paper records a labeled diagram, edge set, nonincreasing degree sequence, a representative minimum secure set for each variant, the three parameter values, and an explicit defender certificate. The values were obtained by exhaustive subset enumeration and reproduced by an independently written bitset implementation.

The classification yields ninety exact parameter values. Four complete graphs satisfy $\gamma_s = \gamma_{sc} < \gamma_{st}$; four noncomplete graphs satisfy $\gamma_s = \gamma_{st} = \gamma_{sc} = 2$; and the remaining twenty-two graphs satisfy $\gamma_s < \gamma_{st} = \gamma_{sc}$. Thus, among noncomplete connected graphs of order at most five, secure total and secure connected domination coincide throughout the census. The thirty graphs realize twenty-eight-degree sequences. The two repeated degree sequences occur in non-isomorphic pairs, but each pair has the same parameter triple; consequently, the degree sequence determines the triple within this finite corpus, although not in general. General results proved in the paper include the basic parameter inequalities, the ordering $\gamma_{st}(G) \leq \gamma_{sc}(G)$ for noncomplete connected graphs, exact values for complete graphs and stars, and a characterization of graphs with $\gamma_{st}(G) = 2$. The resulting article is intended as a transparent small-order reference, a source of worked examples, and a verified base for later structural conjectures.

Keywords: *secure domination; secure total domination; secure connected domination; degree sequence; graph atlas; exhaustive enumeration; defender certificate; small graphs*

2020 Mathematics Subject Classification: *05C69; 05C85*

Research highlights

- Complete classification of all 30 connected non-isomorphic graphs with $2 \leq |V(G)| \leq 5$.
- All 30 labeled graph diagrams, degree sequences, witness sets, and parameter triples are reported.
- Explicit defender certificates make every claimed secure set directly checkable.
- Two independent exact implementations agree on all 90 parameter values.
- Structural theorems explain the complete-graph, universal-vertex, and star cases.

1. Introduction

Domination in graphs provides a natural language for placing monitors, service units, or guards so that every location is covered. A dominating set is static: it certifies coverage at the moment of placement. Secure domination adds a controlled movement requirement. When a vertex outside the selected set is attacked, a neighboring selected vertex must be able to move to it, and the new set must still dominate the graph. This exchange condition was formalized in the protection model of Cockayne et al. [2] and subsequently developed in several directions [3-7].

The present paper compares three exchange-based parameters. The secure domination number $\gamma_s(G)$ uses ordinary domination as the preserved property. The secure total domination number $\gamma_{st}(G)$ requires every vertex, including every selected vertex, to have a selected neighbor before

and after the exchange. The secure connected domination number $\gamma_{sc}(G)$ requires the selected vertices to induce a connected subgraph before and after the exchange. These variants are close enough to invite comparison, but their feasible sets need not coincide.

Large computational studies can be useful, but a table of unexplained values is difficult to audit. A small-order classification has a different purpose: every graph can be displayed, every degree can be checked from the diagram, every proposed minimum set can be inspected, and every defender exchange can be written down.

This transparency is particularly valuable when a paper is intended to support teaching, viva examination, or the development of conjectures.

We therefore study all connected non-isomorphic simple graphs of order at most five, excluding order one because K_1 has no total dominating set. The resulting corpus contains

$$1 + 2 + 6 + 21 = 30$$

graphs of orders 2,3,4, and 5, respectively. The principal research questions are as follows.

1. What are $\gamma_s(G)$, $\gamma_{st}(G)$, and $\gamma_{sc}(G)$ for every graph in this finite corpus?
2. Which parameter orderings occur, and which do not occur, up to order five?
3. How do degree sequences, universal vertices, leaves, and connectivity requirements help explain the observed triples?
4. Can the computation be accompanied by certificates that make every upper bound directly verifiable and every minimum reproducible?

The contribution is not merely a list of values. The paper combines formal results, complete diagrams, representative minimum sets, graph-level defender certificates, independent computational verification, and detailed worked examples. Claims about the thirty-graph corpus are exact; broader claims are stated only when proved.

2. Literature review and research position

The standard foundations of domination are developed in the monograph of Haynes, Hedetniemi and Slater [1]. Secure domination emerged from the graph-protection model [2], while secure total domination [3,4] and secure connected domination [5] introduced stronger preserved properties. Early work established definitions, relationships, bounds, and examples; later work increasingly emphasized algorithms, complexity, products, geometric intersection graphs, and restricted graph classes.

Table 1 summarizes recent literature that directly informs the scope of this paper. The review concentrates on work published from 2017 through 2025, while the foundational sources are retained in the reference list.

Table 1. Recent literature relevant to secure domination variants.

Year	Study	Main focus	Relevance to the present paper
2017	Duginov [8]	Secure total domination: bounds and computational complexity.	Motivates exact small-order baselines for a generally difficult optimization problem.
2017	Li, Shao and Xu [9]	Secure domination in trees.	Shows how structural restrictions can convert a broad problem into exact formulas.
2018	Araki and Miyazaki [10]	Secure domination in proper interval graphs.	Illustrates polynomial-time solvability on a special graph class.
2019	Jha, Pradhan and Banerjee [11]	The secure domination problem in cographs.	Provides a graph-class algorithmic perspective and supports explicit structural classification.

Year	Study	Main focus	Relevance to the present paper
2019	Cabrera Martínez, Montejano and Rodríguez-Velázquez [12]	Relations and bounds for the secure total domination number.	Supports comparison of secure and secure-total parameters.
2020	Kumar, Reddy and Arumugam [13]	Algorithmic complexity of secure connected domination.	Establishes the relevance of treating connected security as a distinct parameter.
2020	Cabrera Martínez, Estrada-Moreno and Rodríguez-Velázquez [14]	Secure total domination in rooted products.	Demonstrates the role of graph construction in exact parameter analysis.
2021	Hernández-Ortiz, Montejano and Rodríguez-Velázquez [15]	Secure domination in rooted product graphs.	Provides a parallel product-graph direction for ordinary secure domination.
2022	Poureidi [16]	Algorithmic results for secure total dominating sets.	Reinforces the need for reproducible computation and graph-class-aware analysis.
2022	Haythorpe and Newcombe [17]	Secure domination in Cartesian products with paths and cycles.	Shows how finite calculations can lead to formulas and conjectures for graph families.
2023	Wang, Yang and Xu [18,19]	Secure, secure-total and secure-connected domination in geometric intersection graphs.	Places the three variants in a common algorithmic setting.
2024	Araki and Aita; Aita and Araki [20,21]	Secure total domination in proper interval and maximal outerplanar graphs.	Recent examples of exact and algorithmic treatment on restricted classes.
2024	Araki; Li and Zhang [22,23]	Bounds for outerplanar graphs and approximation schemes for planar/growth-bounded graphs.	Shows continuing interest in structural bounds and approximation.

The literature shows substantial progress on individual variants and structured graph classes. What is less common is a fully explicit common census in which the same labeled graphs are tested under all three definitions and the result for each graph is accompanied by a witness and exchange certificate. The present paper fills that finite-data and exposition gap. It does not compete with large-scale complexity results; instead, it provides a small, transparent laboratory in which the definitions and their differences can be seen directly.

3. Definitions and notation

All graphs are finite, simple, undirected, and connected. Let $G = (V, E)$ have order $n = |V(G)|$ and size $m = |E(G)|$. The open neighborhood of v is $N(v)$, and $N[v] = N(v) \cup \{v\}$ is its closed neighborhood. The degree of v is $d(v) = |N(v)|$. The degree sequence is written in nonincreasing order,

$$\pi(G) = (d_1, d_2, \dots, d_n), \qquad d_1 \geq d_2 \geq \cdots \geq d_n.$$

A vertex of degree $n - 1$ is universal. For $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$.

Definition 3.1 (dominating set). A set $S \subseteq V(G)$ is dominating if every vertex of $V(G) - S$ has a neighbor in S.

Definition 3.2 (total dominating set). A set $S \subseteq V(G)$ is total dominating if every vertex of G has a neighbor in S. In particular, a selected vertex cannot dominate itself.

Definition 3.3 (connected dominating set). A set $S \subseteq V(G)$ is connected dominating if it is dominating and $G[S]$ is connected.

For $u \notin S$ and $v \in S \cap N(u)$, define the exchange set

$$S_{v \to u} = (S - \{v\}) \cup \{u\}.$$

The vertex v is called a defender of u with respect to a required base property when $S_{v \to u}$ still has that property.

Definition 3.4 (secure dominating set). A dominating set S is secure if every $u \in V(G) - S$ has a defender $v \in S \cap N(u)$ such that $S_{v \to u}$ is a dominating set. The minimum cardinality of such a set is $\gamma_s(G)$.

Definition 3.5 (secure total dominating set). A total dominating set S is secure total if every $u \in V(G) - S$ has a defender $v \in S \cap N(u)$ such that $S_{v \to u}$ is total dominating. Its minimum cardinality is $\gamma_{st}(G)$.

Definition 3.6 (secure connected dominating set). A connected dominating set S is secure connected if every $u \in V(G) - S$ has a defender $v \in S \cap N(u)$ such that $S_{v \to u}$ is connected dominating. Its minimum cardinality is $\gamma_{sc}(G)$.

A **defender certificate** for a secure set S lists, for every $u \notin S$, at least one valid defender v and the exchanged set $S_{v \to u}$. Such a certificate proves the corresponding upper bound. Minimality is established either by a structural lower-bound argument or by exhaustive rejection of all smaller subsets.

4. Graph corpus and verification method

4.1 Complete non-isomorphic corpus

The graph list was obtained from the NetworkX implementation of the Read-Wilson Graph Atlas [26,27]. The Atlas contains one representative of every isomorphism class of graphs with at most seven vertices. We retained precisely the connected graphs with $2 \leq n \leq 5$ and relabeled their vertices as $1,2,\ldots,n$. Within each order, the graphs are indexed in Atlas order and denoted by $G_{n,i}$. This produces the counts shown in Table 2.

Table 2. Number of connected non-isomorphic graphs in the corpus.

Order n	Number of graphs	Identifiers
2	1	$G_{2,1}$
3	2	$G_{3,1}$ to $G_{3,2}$
4	6	$G_{4,1}$ to $G_{4,6}$
5	21	$G_{5,1}$ to $G_{5,21}$
Total	30	

The identifiers are labels for this paper, not canonical graph names. The edge set and diagram supplied for each identifier remove any ambiguity.

4.2 Exact search

For each graph and each base property $P \in$ {dominating,total dominating,connected dominating}, subsets were examined in increasing cardinality. A candidate S was accepted only if it had property P and every outside vertex had a valid defender preserving P. The first accepted cardinality is therefore exact.

Algorithm 1. Exact secure-set search for a base property P.

1. For $k = 1,2,\ldots,n$, enumerate every k-subset $S \subseteq V(G)$.
2. Reject S if S does not have property P.
3. For each $u \in V(G) - S$, search $S \cap N(u)$ for a vertex v such that $S_{v \to u}$ has property P.
4. Reject S if any outside vertex lacks a valid defender.
5. Return the first cardinality k for which at least one set survives, together with all minimum survivors and their certificates.

The corpus is small enough for direct exhaustive enumeration. A straightforward implementation has a worst-case subset factor of 2^n; the remaining checks are polynomial in n and m. For $n \leq 5$, the emphasis is not speed but traceability.

4.3 Independent verification

Two implementations were used. The first used NetworkX graph objects and Python sets. The second represented adjacency and selected sets as integer bit masks and used separate routines for domination, total domination, induced connectivity, and exchange testing. The implementations do not share the same internal representation or base-property code. They agreed on all $30 \times 3 = 90$ minimum values and on the complete collections of minimum sets.

Additional consistency checks were applied: degree sums equal $2m$ for every graph; complete graphs satisfy the formulas proved in Section 5; stars satisfy the formulas in Theorem 5.5; and all displayed certificates were regenerated from the stored graph data. The CSV file, JSON certificate file, source scripts, and figures are included in the supplementary archive.

5. General results

The following facts explain much of the finite classification.

Proposition 5.1. For every connected graph G for which the parameters are defined,

$$\gamma_s(G) \leq \gamma_{st}(G) \quad \text{and} \quad \gamma_s(G) \leq \gamma_{sc}(G).$$

Proof. Every total dominating set is dominating, and every connected dominating set is dominating. If an exchange preserves total domination or connected domination, then it also preserves ordinary domination. Hence every STDS and every SCDS is an SDS. Taking minimum cardinalities gives the inequalities. ▫

Lemma 5.2. A connected graph has a secure dominating set of cardinality one if and only if it is complete. The same statement holds for secure connected domination.

Proof. If $S = \{v\}$ is secure dominating, then v dominates G. For every $u \neq v$, the only possible exchange produces $\{u\}$, which must also dominate G. Thus, every vertex is universal and G is complete. Conversely, every singleton in a complete graph is secure dominating and induces a connected subgraph. ▫

Theorem 5.3. If G is a noncomplete connected graph, then

$$\gamma_s(G) \leq \gamma_{st}(G) \leq \gamma_{sc}(G).$$

Proof. The first inequality is Proposition 5.1. Let S be an SCDS. Because G is noncomplete, Lemma 5.2 implies $|S| \geq 2$. Since $G[S]$ is connected and has at least two vertices, every vertex of S has a neighbor in S; domination supplies a selected neighbor for every vertex outside S. Hence S is total dominating. For each required exchange, $S_{v \to u}$ is connected dominating and has the same cardinality at least two, so it is also total dominating. Thus every SCDS is an STDS, and $\gamma_{st}(G) \leq \gamma_{sc}(G)$. ▫

Proposition 5.4. For the complete graph K_n with $n \geq 2$,

$$\big(\gamma_s(K_n), \gamma_{st}(K_n), \gamma_{sc}(K_n)\big) = (1,2,1).$$

Proof. Every singleton dominates a complete graph, remains a singleton after an exchange, and induces a connected subgraph. Hence $\gamma_s(K_n) = \gamma_{sc}(K_n) = 1$. A total dominating set cannot have size one. Any two vertices form a secure total dominating set, so $\gamma_{st}(K_n) = 2$. ▫

Theorem 5.5. Let $m \geq 2$. For the star $K_{1,m}$,

$$\big(\gamma_s(K_{1,m}), \gamma_{st}(K_{1,m}), \gamma_{sc}(K_{1,m})\big) = (m, m+1, m+1).$$

Proof. Let c be the center and L the set of m leaves. The set L is an SDS: the only outside vertex is c, and replacing any leaf by c leaves a dominating set. Thus $\gamma_s \leq m$. If an SDS omits c, domination forces all leaves into the set. If it contains c and omits two leaves, then exchanging c for either omitted leaf leaves the other omitted leaf undominated. Hence every SDS has at least m vertices.

Every total dominating set of the star contains c and at least one leaf. If a leaf u is outside an STDS, its only possible defender is c; after replacing c by u, the selected leaves have no selected neighbors. Therefore, an STDS contains every vertex. The same endpoint exchange destroys connected domination for every proper connected dominating set. Consequently $\gamma_{st} = \gamma_{sc} = m + 1$. ▫

Theorem 5.6. A graph G without isolated vertices satisfies $\gamma_{st}(G) = 2$ if and only if G has at least two universal vertices.

Proof. Suppose $S = \{x, y\}$ is an STDS. Total domination gives $xy \in E(G)$. Let $u \notin S$. If x defends u, then $ux \in E(G)$ and the exchanged pair $\{u, y\}$ is total dominating, so $uy \in E(G)$. If y defends u, the same conclusion follows with the roles reversed. Thus, every outside vertex is adjacent to both x and y, and x, y are universal. Conversely, let x, y be universal. Then $\{x, y\}$ is total dominating. For every $u \notin \{x, y\}$, replacing x by u gives the adjacent pair $\{u, y\}$; the universal vertex y dominates every other vertex and is itself adjacent to u. Hence the exchanged pair is total dominating. ▫

Corollary 5.7. If G is noncomplete and has at least two universal vertices, then

$$\gamma_s(G) = \gamma_{st}(G) = \gamma_{sc}(G) = 2.$$

Proof. Theorem 5.6 supplies an STDS of size two, and the same two universal vertices form an SCDS. Proposition 5.1 gives upper bounds of two for all three parameters. Lemma 5.2 excludes value one in a noncomplete graph. ▫

6. Complete atlas of the thirty graphs

Each panel below displays the graph identifier, labeled diagram, nonincreasing degree sequence $\pi(G)$, and parameter triple $(\gamma_s, \gamma_{st}, \gamma_{sc})$. The drawings use the same labels as the edge sets, tables, and certificates.

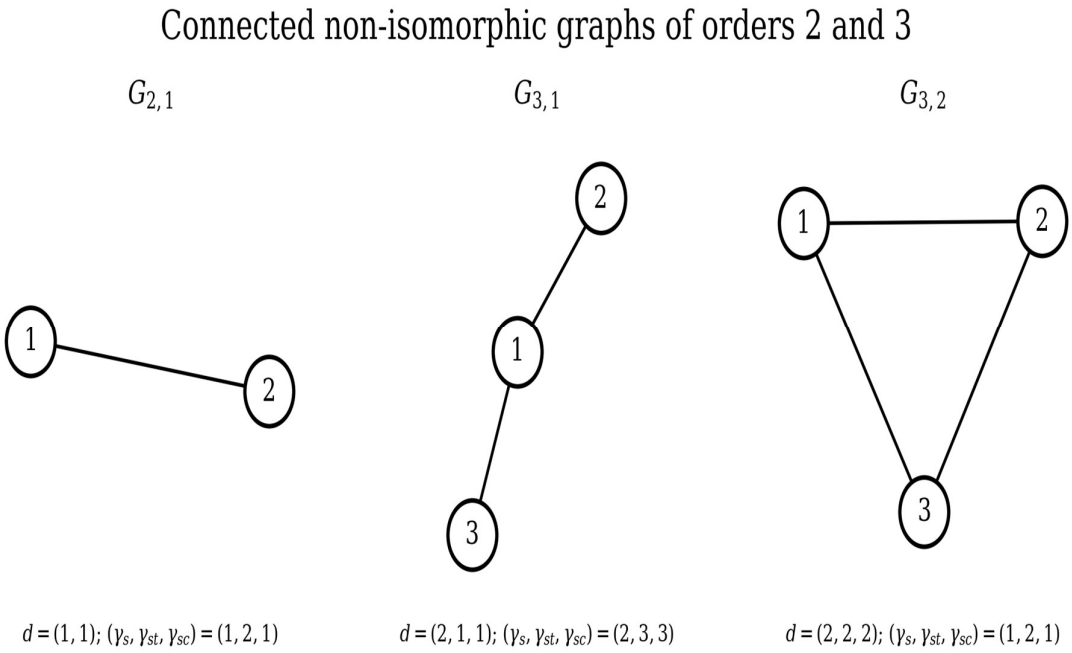


Figure 1. The three connected non-isomorphic graphs of orders two and three.

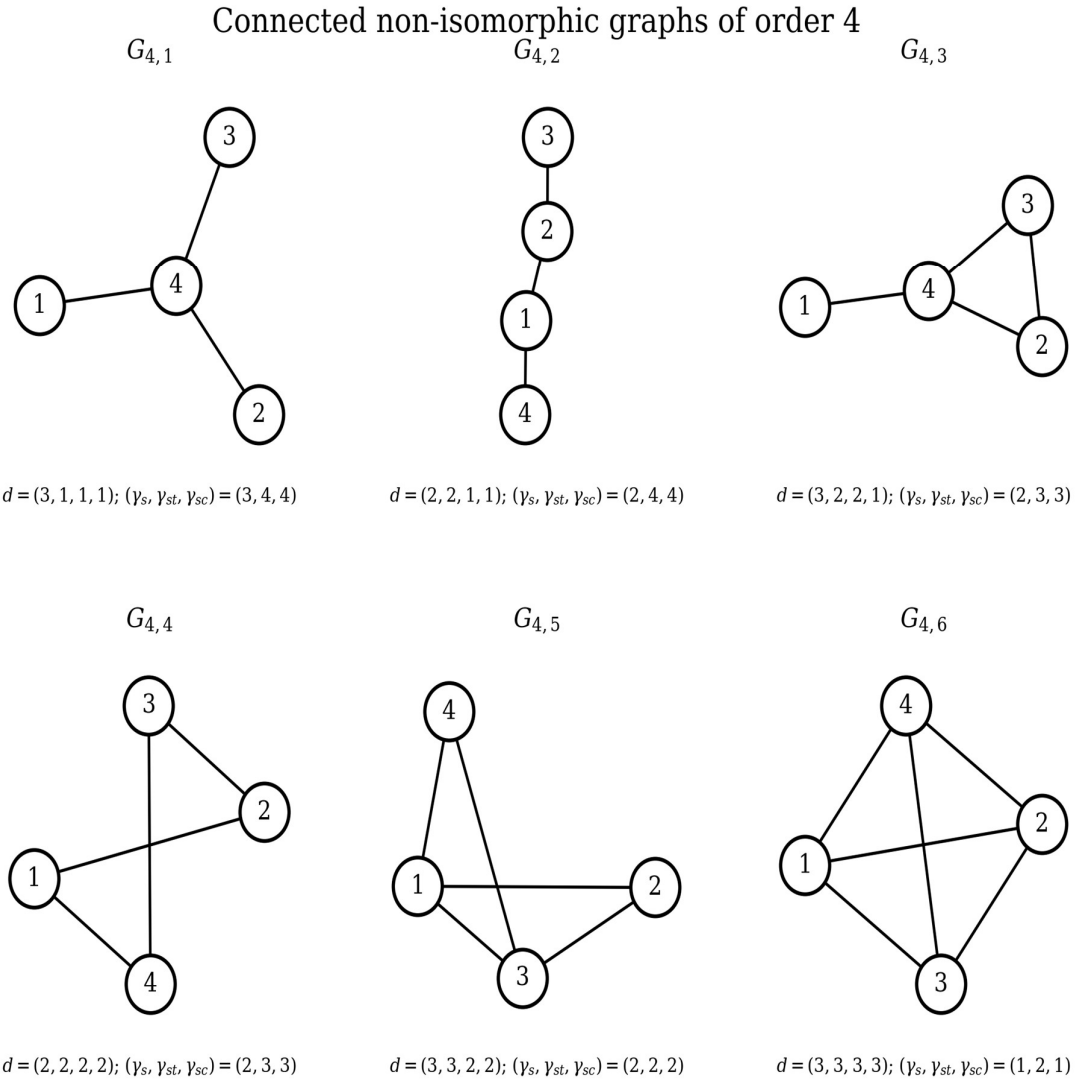
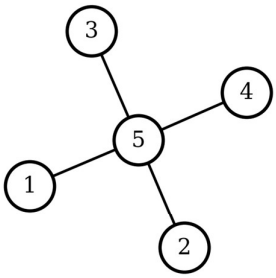


Figure 2. The six connected non-isomorphic graphs of order four.

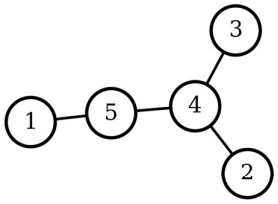
Connected non-isomorphic graphs of order 5 (plate 1 of 3)

$G_{5,1}$



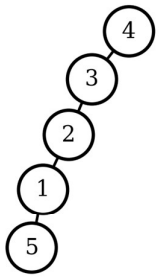
$d = (4, 1, 1, 1, 1)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (4, 5, 5)$

$G_{5,2}$



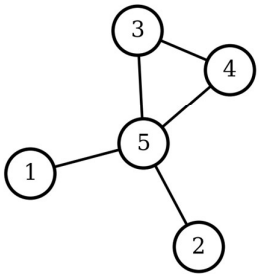
$d = (3, 2, 1, 1, 1)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (3, 5, 5)$

$G_{5,3}$



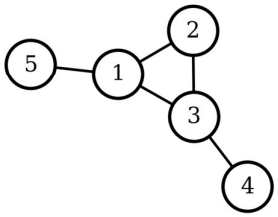
$d = (2, 2, 2, 1, 1)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (3, 5, 5)$

$G_{5,4}$



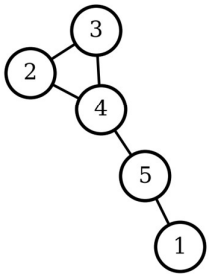
$d = (4, 2, 2, 1, 1)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (3, 4, 4)$

$G_{5,5}$



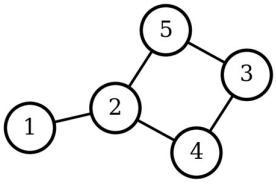
$d = (3, 3, 2, 1, 1)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (3, 5, 5)$

$G_{5,6}$



$d = (3, 2, 2, 2, 1)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (2, 4, 4)$

$G_{5,7}$

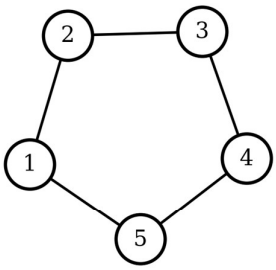


$d = (3, 2, 2, 2, 1)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (2, 4, 4)$

Figure 3. Order-five graphs $G_{5,1}$ through $G_{5,7}$.

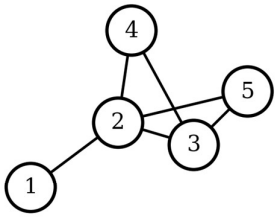
Connected non-isomorphic graphs of order 5 (plate 2 of 3)

$G_{5,8}$



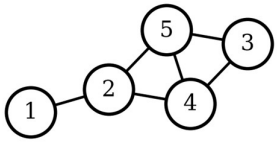
$d = (2, 2, 2, 2, 2)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (3, 4, 4)$

$G_{5,9}$



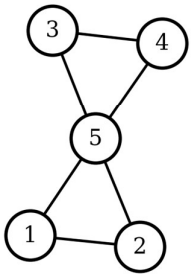
$d = (4, 3, 2, 2, 1)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (2, 3, 3)$

$G_{5,10}$



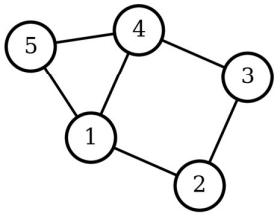
$d = (3, 3, 3, 2, 1)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (2, 4, 4)$

$G_{5,11}$



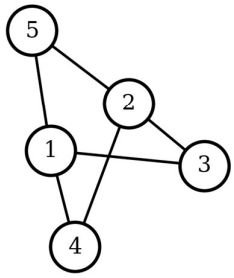
$d = (4, 2, 2, 2, 2)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (2, 3, 3)$

$G_{5,12}$



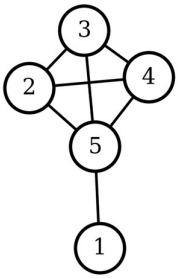
$d = (3, 3, 2, 2, 2)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (2, 3, 3)$

$G_{5,13}$



$d = (3, 3, 2, 2, 2)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (2, 3, 3)$

$G_{5,14}$

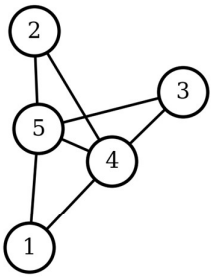


$d = (4, 3, 3, 3, 1)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (2, 3, 3)$

Figure 4. Order-five graphs $G_{5,8}$ through $G_{5,14}$.

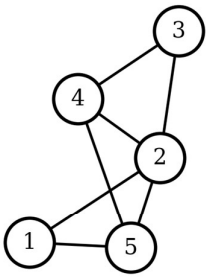
Connected non-isomorphic graphs of order 5 (plate 3 of 3)

$G_{5,15}$



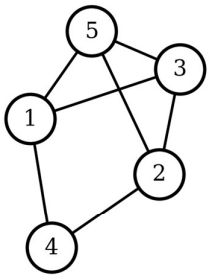
$d = (4, 4, 2, 2, 2)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (2, 2, 2)$

$G_{5,16}$



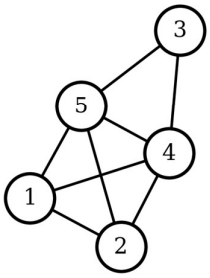
$d = (4, 3, 3, 2, 2)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (2, 3, 3)$

$G_{5,17}$



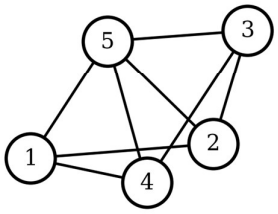
$d = (3, 3, 3, 3, 2)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (2, 3, 3)$

$G_{5,18}$



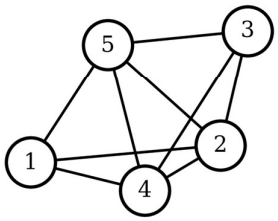
$d = (4, 4, 3, 3, 2)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (2, 2, 2)$

$G_{5,19}$



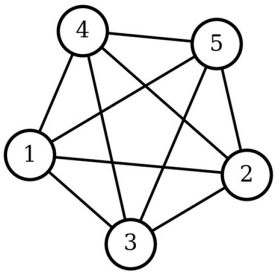
$d = (4, 3, 3, 3, 3)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (2, 3, 3)$

$G_{5,20}$



$d = (4, 4, 4, 3, 3)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (2, 2, 2)$

$G_{5,21}$



$d = (4, 4, 4, 4, 4)$; $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (1, 2, 1)$

Figure 5. Order-five graphs $G_{5,15}$ through $G_{5,21}$.

7. Complete structural and parameter classification

7.1 Structural data

Tables 3-6 give the edge set, degree sequence, and universal vertices for every graph. An edge uv denotes the unordered pair $\{u, v\}$.

Table 3. Structural data for connected graphs of order **2**.

Graph	n	m	Degree sequence	Edge set	Universal vertices
$G_{2,1}$	2	1	(1,1)	{12}	{1,2}

Table 4. Structural data for connected graphs of order **3**.

Graph	n	m	Degree sequence	Edge set	Universal vertices
$G_{3,1}$	3	2	(2,1,1)	{12,13}	{1}
$G_{3,2}$	3	3	(2,2,2)	{12,13,23}	{1,2,3}

Table 5. Structural data for connected graphs of order **4**.

Graph	n	m	Degree sequence	Edge set	Universal vertices
$G_{4,1}$	4	3	(3,1,1,1)	{14,24,34}	{4}
$G_{4,2}$	4	3	(2,2,1,1)	{12,14,23}	⌀
$G_{4,3}$	4	4	(3,2,2,1)	{14,23,24,34}	{4}
$G_{4,4}$	4	4	(2,2,2,2)	{12,14,23,34}	⌀
$G_{4,5}$	4	5	(3,3,2,2)	{12,13,14,23,34}	{1,3}
$G_{4,6}$	4	6	(3,3,3,3)	{12,13,14,23,24,34}	{1,2,3,4}

Table 6. Structural data for connected graphs of order **5**.

Graph	n	m	Degree sequence	Edge set	Universal vertices
$G_{5,1}$	5	4	(4,1,1,1,1)	{15,25,35,45}	{5}
$G_{5,2}$	5	4	(3,2,1,1,1)	{15,24,34,45}	⌀
$G_{5,3}$	5	4	(2,2,2,1,1)	{12,15,23,34}	⌀
$G_{5,4}$	5	5	(4,2,2,1,1)	{15,25,34,35,45}	{5}
$G_{5,5}$	5	5	(3,3,2,1,1)	{12,13,15,23,34}	⌀
$G_{5,6}$	5	5	(3,2,2,2,1)	{15,23,24,34,45}	⌀
$G_{5,7}$	5	5	(3,2,2,2,1)	{12,24,25,34,35}	⌀
$G_{5,8}$	5	5	(2,2,2,2,2)	{12,15,23,34,45}	⌀
$G_{5,9}$	5	6	(4,3,2,2,1)	{12,23,24,25,34,35}	{2}
$G_{5,10}$	5	6	(3,3,3,2,1)	{12,24,25,34,35,45}	⌀
$G_{5,11}$	5	6	(4,2,2,2,2)	{12,15,25,34,35,45}	{5}
$G_{5,12}$	5	6	(3,3,2,2,2)	{12,14,15,23,34,45}	⌀

Graph	n	m	Degree sequence	Edge set	Universal vertices
$G_{5,13}$	5	6	(3,3,2,2,2)	{13,14,15,23,24,25}	⌀
$G_{5,14}$	5	7	(4,3,3,3,1)	{15,23,24,25,34,35,45}	{5}
$G_{5,15}$	5	7	(4,4,2,2,2)	{14,15,24,25,34,35,45}	{4,5}
$G_{5,16}$	5	7	(4,3,3,2,2)	{12,15,23,24,25,34,45}	{2}
$G_{5,17}$	5	7	(3,3,3,3,2)	{13,14,15,23,24,25,35}	⌀
$G_{5,18}$	5	8	(4,4,3,3,2)	{12,14,15,24,25,34,35,45}	{4,5}
$G_{5,19}$	5	8	(4,3,3,3,3)	{12,14,15,23,25,34,35,45}	{5}
$G_{5,20}$	5	9	(4,4,4,3,3)	{12,14,15,23,24,25,34,35,45}	{2,4,5}
$G_{5,21}$	5	10	(4,4,4,4,4)	{12,13,14,15,23,24,25,34,35,45}	{1,2,3,4,5}

7.2 Parameter values and representative minimum sets

Tables 7-10 report one representative minimum set for each variant. The set written after a parameter value is a witness under the displayed vertex labeling; Appendix A gives the defender certificate and the number of minimum witnesses for every graph.

Table 7. Secure-domination parameters for connected graphs of order **2**.

Graph	Degree sequence	m	Secure domination	Secure total domination	Secure connected domination	Pattern
$G_{2,1}$	(1,1)	1	$\gamma_s = 1, \{1\}$	$\gamma_{st} = 2, \{1,2\}$	$\gamma_{sc} = 1, \{1\}$	$\gamma_s = \gamma_{sc} < \gamma_{st}$

Table 8. Secure-domination parameters for connected graphs of order **3**.

Graph	Degree sequence	m	Secure domination	Secure total domination	Secure connected domination	Pattern
$G_{3,1}$	(2,1,1)	2	$\gamma_s = 2, \{1,2\}$	$\gamma_{st} = 3, \{1,2,3\}$	$\gamma_{sc} = 3, \{1,2,3\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{3,2}$	(2,2,2)	3	$\gamma_s = 1, \{1\}$	$\gamma_{st} = 2, \{1,2\}$	$\gamma_{sc} = 1, \{1\}$	$\gamma_s = \gamma_{sc} < \gamma_{st}$

Table 9. Secure-domination parameters for connected graphs of order **4**.

Graph	Degree sequence	m	Secure domination	Secure total domination	Secure connected domination	Pattern
$G_{4,1}$	(3,1,1,1)	3	$\gamma_s = 3, \{1,2,3\}$	$\gamma_{st} = 4, \{1,2,3,4\}$	$\gamma_{sc} = 4, \{1,2,3,4\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{4,2}$	(2,2,1,1)	3	$\gamma_s = 2, \{1,2\}$	$\gamma_{st} = 4, \{1,2,3,4\}$	$\gamma_{sc} = 4, \{1,2,3,4\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{4,3}$	(3,2,2,1)	4	$\gamma_s = 2, \{1,2\}$	$\gamma_{st} = 3, \{1,2,4\}$	$\gamma_{sc} = 3, \{1,2,4\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{4,4}$	(2,2,2,2)	4	$\gamma_s = 2, \{1,2\}$	$\gamma_{st} = 3, \{1,2,3\}$	$\gamma_{sc} = 3, \{1,2,3\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{4,5}$	(3,3,2,2)	5	$\gamma_s = 2, \{1,2\}$	$\gamma_{st} = 2, \{1,3\}$	$\gamma_{sc} = 2, \{1,3\}$	$\gamma_s = \gamma_{st} = \gamma_{sc}$

Graph	Degree sequence	m	Secure domination	Secure total domination	Secure connected domination	Pattern
$G_{4,6}$	(3,3,3,3)	6	$\gamma_s = 1, \{1\}$	$\gamma_{st} = 2, \{1,2\}$	$\gamma_{sc} = 1, \{1\}$	$\gamma_s = \gamma_{sc} < \gamma_{st}$

Table 10. Secure-domination parameters for connected graphs of order **5**.

Graph	Degree sequence	m	Secure domination	Secure total domination	Secure connected domination	Pattern
$G_{5,1}$	(4,1,1,1,1)	4	$\gamma_s = 4, \{1,2,3,4\}$	$\gamma_{st} = 5, \{1,2,3,4,5\}$	$\gamma_{sc} = 5, \{1,2,3,4,5\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,2}$	(3,2,1,1,1)	4	$\gamma_s = 3, \{1,2,3\}$	$\gamma_{st} = 5, \{1,2,3,4,5\}$	$\gamma_{sc} = 5, \{1,2,3,4,5\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,3}$	(2,2,2,1,1)	4	$\gamma_s = 3, \{1,2,3\}$	$\gamma_{st} = 5, \{1,2,3,4,5\}$	$\gamma_{sc} = 5, \{1,2,3,4,5\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,4}$	(4,2,2,1,1)	5	$\gamma_s = 3, \{1,2,3\}$	$\gamma_{st} = 4, \{1,2,3,5\}$	$\gamma_{sc} = 4, \{1,2,3,5\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,5}$	(3,3,2,1,1)	5	$\gamma_s = 3, \{1,2,3\}$	$\gamma_{st} = 5, \{1,2,3,4,5\}$	$\gamma_{sc} = 5, \{1,2,3,4,5\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,6}$	(3,2,2,2,1)	5	$\gamma_s = 2, \{1,2\}$	$\gamma_{st} = 4, \{1,2,3,5\}$	$\gamma_{sc} = 4, \{1,2,4,5\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,7}$	(3,2,2,2,1)	5	$\gamma_s = 2, \{2,3\}$	$\gamma_{st} = 4, \{1,2,3,4\}$	$\gamma_{sc} = 4, \{1,2,3,4\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,8}$	(2,2,2,2,2)	5	$\gamma_s = 3, \{1,2,3\}$	$\gamma_{st} = 4, \{1,2,3,4\}$	$\gamma_{sc} = 4, \{1,2,3,4\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,9}$	(4,3,2,2,1)	6	$\gamma_s = 2, \{2,3\}$	$\gamma_{st} = 3, \{1,2,3\}$	$\gamma_{sc} = 3, \{1,2,3\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,10}$	(3,3,3,2,1)	6	$\gamma_s = 2, \{1,3\}$	$\gamma_{st} = 4, \{1,2,3,4\}$	$\gamma_{sc} = 4, \{1,2,3,4\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,11}$	(4,2,2,2,2)	6	$\gamma_s = 2, \{1,3\}$	$\gamma_{st} = 3, \{1,3,5\}$	$\gamma_{sc} = 3, \{1,3,5\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,12}$	(3,3,2,2,2)	6	$\gamma_s = 2, \{1,2\}$	$\gamma_{st} = 3, \{1,2,4\}$	$\gamma_{sc} = 3, \{1,2,4\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,13}$	(3,3,2,2,2)	6	$\gamma_s = 2, \{1,2\}$	$\gamma_{st} = 3, \{1,2,3\}$	$\gamma_{sc} = 3, \{1,2,3\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,14}$	(4,3,3,3,1)	7	$\gamma_s = 2, \{1,2\}$	$\gamma_{st} = 3, \{1,2,5\}$	$\gamma_{sc} = 3, \{1,2,5\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,15}$	(4,4,2,2,2)	7	$\gamma_s = 2, \{4,5\}$	$\gamma_{st} = 2, \{4,5\}$	$\gamma_{sc} = 2, \{4,5\}$	$\gamma_s = \gamma_{st} = \gamma_{sc}$
$G_{5,16}$	(4,3,3,2,2)	7	$\gamma_s = 2, \{1,2\}$	$\gamma_{st} = 3, \{1,2,3\}$	$\gamma_{sc} = 3, \{1,2,3\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,17}$	(3,3,3,3,2)	7	$\gamma_s = 2, \{1,2\}$	$\gamma_{st} = 3, \{1,2,3\}$	$\gamma_{sc} = 3, \{1,2,3\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,18}$	(4,4,3,3,2)	8	$\gamma_s = 2, \{1,3\}$	$\gamma_{st} = 2, \{4,5\}$	$\gamma_{sc} = 2, \{4,5\}$	$\gamma_s = \gamma_{st} = \gamma_{sc}$

Graph	Degree sequence	m	Secure domination	Secure total domination	Secure connected domination	Pattern
$G_{5,19}$	(4,3,3,3,3)	8	$\gamma_s = 2, \{1,2\}$	$\gamma_{st} = 3, \{1,2,3\}$	$\gamma_{sc} = 3, \{1,2,3\}$	$\gamma_s < \gamma_{st} = \gamma_{sc}$
$G_{5,20}$	(4,4,4,3,3)	9	$\gamma_s = 2, \{1,2\}$	$\gamma_{st} = 2, \{2,4\}$	$\gamma_{sc} = 2, \{2,4\}$	$\gamma_s = \gamma_{st} = \gamma_{sc}$
$G_{5,21}$	(4,4,4,4,4)	10	$\gamma_s = 1, \{1\}$	$\gamma_{st} = 2, \{1,2\}$	$\gamma_{sc} = 1, \{1\}$	$\gamma_s = \gamma_{sc} < \gamma_{st}$

7.3 Distribution of comparison patterns

The observed comparison patterns are summarized in Table 11 and Figure 6.

Table 11. Distribution of parameter relations across the thirty graphs.

Parameter relation	Number of graphs	Percentage	Interpretation
$\gamma_s = \gamma_{st} = \gamma_{sc}$	4	13.33%	All three minimum requirements coincide.
$\gamma_s = \gamma_{sc} < \gamma_{st}$	4	13.33%	Complete-graph pattern in the present corpus.
$\gamma_s < \gamma_{st} = \gamma_{sc}$	22	73.33%	The dominant pattern for noncomplete graphs.
$\gamma_s < \gamma_{st} < \gamma_{sc}$	0	0.00%	No example occurs for order at most five.
$\gamma_s = \gamma_{st} < \gamma_{sc}$	0	0.00%	No example occurs for order at most five.

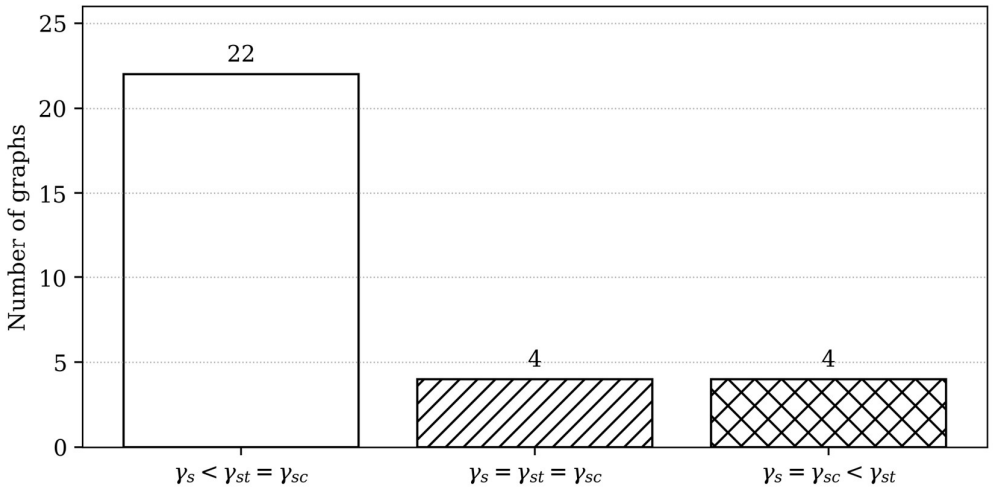


Figure 6. Number of graphs in each realized comparison class.

The complete graphs K_2, K_3, K_4, and K_5 account for the four cases $\gamma_s = \gamma_{sc} < \gamma_{st}$. Four dense noncomplete graphs have at least two universal vertices and therefore satisfy $\gamma_s = \gamma_{st} = \gamma_{sc} = 2$ by Corollary 5.7. Every other graph satisfies $\gamma_s < \gamma_{st} = \gamma_{sc}$. In particular, no graph of order at most five realizes $\gamma_s < \gamma_{st} < \gamma_{sc}$ or $\gamma_s = \gamma_{st} < \gamma_{sc}$.

8. Detailed graph-by-graph calculations for representative models

This section illustrates the reasoning behind the tables. The appendix supplies compact certificates for all thirty graphs.

8.1 The complete graph $G_{2,1} = K_2$

The degree sequence is (1,1). A singleton, for example $S = \{1\}$, dominates K_2; the only outside vertex 2 is defended by 1, and the exchange gives {2}. The same singleton is connected. No singleton is total dominating because its selected vertex has no selected neighbor. Hence

$$\left(\gamma_s(G_{2,1}), \gamma_{st}(G_{2,1}), \gamma_{sc}(G_{2,1})\right) = (1,2,1).$$

8.2 The path $G_{3,1} = P_3$

The edges are 12 and 13, so vertex 1 is the center and the degree sequence is (2,1,1). The set {1,2} is secure dominating. Its only outside vertex is 3, defended by 1; the exchange {2,3} dominates the center. No singleton is secure by Lemma 5.2. The star formula with $m = 2$ gives

$$\left(\gamma_s(G_{3,1}), \gamma_{st}(G_{3,1}), \gamma_{sc}(G_{3,1})\right) = (2,3,3).$$

8.3 The path $G_{4,2} = P_4$

Here $E(G_{4,2}) = \{12,14,23\}$, so the path order is $3 - 2 - 1 - 4$ and $\pi(G_{4,2}) = (2,2,1,1)$. Let $S = \{1,2\}$. Vertex 3 is defended by 2, producing {1,3}, and vertex 4 is defended by 1, producing {2,4}. Both exchanged sets dominate the path. Since the graph is noncomplete, no singleton is secure; therefore $\gamma_s(G_{4,2}) = 2$.

For total domination, the central pair {1,2} is total dominating, but it is not secure total: exchanging 2 for endpoint 3 gives {1,3}, whose selected vertices are not adjacent. Every three-vertex candidate either fails total domination or fails an endpoint exchange. The full vertex set is therefore the unique STDS. The same endpoint exchanges disconnect every proper connected dominating candidate, so the full vertex set is also the unique SCDS. Thus

$$\left(\gamma_s(G_{4,2}), \gamma_{st}(G_{4,2}), \gamma_{sc}(G_{4,2})\right) = (2,4,4).$$

8.4 The diamond $G_{4,5} = K_4 - e$

The degree sequence is (3,3,2,2), and vertices 1 and 3 are universal. The graph is not complete. By Corollary 5.7, two universal vertices give a secure total and secure connected dominating set, while Lemma 5.2 excludes value one. Consequently

$$\left(\gamma_s(G_{4,5}), \gamma_{st}(G_{4,5}), \gamma_{sc}(G_{4,5})\right) = (2,2,2).$$

This graph is the smallest noncomplete example in the census for which all three parameters agree.

8.5 The star $G_{5,1} = K_{1,4}$

Vertex 5 is the center and {1,2,3,4} is the leaf set. Selecting all four leaves yields an SDS: the center is the only outside vertex, and replacing any leaf by the center preserves domination. Theorem 5.5 gives the matching lower bound. For total or connected security, any omitted leaf can be defended only by the center, but moving the center destroys the relevant property. Hence every vertex is required for the two stronger variants:

$$\left(\gamma_s(G_{5,1}), \gamma_{st}(G_{5,1}), \gamma_{sc}(G_{5,1})\right) = (4,5,5).$$

8.6 The path $G_{5,3} = P_5$

The edge set is {12,15,23,34}, giving the path $5 - 1 - 2 - 3 - 4$ and degree sequence (2,2,2,1,1). Take $S = \{1,2,3\}$. Endpoint 4 is defended by 3, producing {1,2,4}; endpoint 5 is defended by 1, producing {2,3,5}. Both sets dominate the path. A direct check of all ten two-vertex subsets shows that none is secure, so $\gamma_s(G_{5,3}) = 3$.

For secure total domination, the full set works. Among four-vertex sets, $V - \{1\}$ and $V - \{3\}$ fail total domination because an endpoint loses its only neighbor. The sets $V - \{4\}$ and $V - \{5\}$ fail when the omitted endpoint is defended, because the exchanged endpoint has no selected neighbor. The remaining set $V - \{2\}$ is total dominating, but neither possible exchange for vertex 2 preserves total domination. Thus $\gamma_{st}(G_{5,3}) = 5$. An analogous check of proper connected

dominating sets shows that an endpoint exchange disconnects the selected subgraph, and therefore $\gamma_{sc}(G_{5,3}) = 5$.

8.7 The cycle $G_{5,8} = C_5$

The graph is 2-regular. The set $S = \{1,2,3\}$ is secure dominating: vertex 4 is defended by 3, giving $\{1,2,4\}$, and vertex 5 is defended by 1, giving $\{2,3,5\}$. Every dominating pair in C_5 leaves an outside vertex for which both possible exchanges fail, so $\gamma_s(C_5) = 3$.

Any four consecutive vertices form both an STDS and an SCDS; after the omitted vertex is inserted, one of its two neighbors can leave while the selected vertices remain total dominating and connected. No three-vertex set is secure total or secure connected. Hence

$$\left(\gamma_s(G_{5,8}), \gamma_{st}(G_{5,8}), \gamma_{sc}(G_{5,8})\right) = (3,4,4).$$

8.8 A dense universal-vertex model: $G_{5,15}$

The degree sequence is $(4,4,2,2,2)$. Vertices 4 and 5 are universal, while the graph is not complete. The pair $\{4,5\}$ is simultaneously secure dominating, secure total dominating, and secure connected dominating. Every outside vertex can replace either universal vertex while the other universal vertex maintains domination, total adjacency, and connectivity. Lemma 5.2 excludes a singleton, so

$$\left(\gamma_s(G_{5,15}), \gamma_{st}(G_{5,15}), \gamma_{sc}(G_{5,15})\right) = (2,2,2).$$

8.9 Two pairs with the same degree sequence

The corpus has twenty-eight distinct degree sequences rather than thirty. The repeated cases are listed in Table 12.

Table 12. Repeated degree sequences within the order-five corpus.

Repeated degree sequence	Non-isomorphic graphs	Parameter triples
$(3,2,2,2,1)$	$G_{5,6}$, $G_{5,7}$	$(2,4,4)$, $(2,4,4)$
$(3,3,2,2,2)$	$G_{5,12}$, $G_{5,13}$	$(2,3,3)$, $(2,3,3)$

The graphs in each row are non-isomorphic because their edge structures differ, even though their degree multisets agree. Their parameter triples also agree. Therefore the degree sequence happens to determine the triple on the present thirty-graph corpus. This is a finite observation, not a general theorem. A boundary search at order six already finds connected graphs with the common degree sequence $(3,2,2,1,1,1)$ but different secure-domination values, namely triples $(3,6,6)$ and $(4,6,6)$. Thus degree data alone cannot replace structural analysis.

9. Discussion

9.1 What the degree sequences explain

Degree sequences immediately identify several decisive structures. The sequence $(n-1, n-1, \ldots, n-1)$ identifies K_n, whose triple is $(1,2,1)$. A star has sequence $(m, 1, \ldots, 1)$ and triple $(m, m+1, m+1)$. A noncomplete graph with at least two entries equal to $n-1$ has at least two universal vertices and therefore triple $(2,2,2)$. These cases account for eight of the thirty graphs and are explained without exhaustive case-by-case exchange testing.

For the remaining graphs, the degree sequence is informative but not decisive. Leaves create fragile exchanges because they have a unique possible defender. Cut vertices and sparse induced subgraphs tend to make connected security expensive. By contrast, high degree creates alternative defenders and often reduces all three parameters. The two repeated degree sequences show that non-isomorphic structure can coexist with the same triple; the order-six boundary example shows that the same degree sequence can eventually produce different triples.

9.2 Equality of the two stronger variants

Theorem 5.3 guarantees $\gamma_{st} \leq \gamma_{sc}$ for noncomplete connected graphs. In the present census, equality holds for all twenty-six noncomplete graphs. This finite regularity should be interpreted cautiously. It suggests that the smallest noncomplete graph satisfying $\gamma_{st} < \gamma_{sc}$ has order at least six, but it does not establish whether such a graph exists at order six or what structural feature first separates the parameters. That is a natural next classification problem.

9.3 Value of explicit certificates

A minimum value has two logical components. A witness set and its defender certificate establish an upper bound. Minimality requires a lower bound. For the named families and universal-vertex cases, the lower bounds are structural. For the remaining finite cases, exhaustive rejection of all smaller subsets supplies a complete proof within the corpus. Presenting both components prevents a common weakness in computational papers: reporting a number without enough information for a reader to verify what the number means.

9.4 Scope and limitations

The results classify exactly the connected non-isomorphic graphs with orders two through five. Frequency statements such as "twenty-two of thirty graphs" describe this corpus only; they are not asymptotic statements about random or large graphs. The Atlas labeling is reproducible but not canonical. The search is exact but exponential, so extension to larger orders requires efficient graph generation, pruning, symmetry handling, or specialized algorithms. Finally, a journal submission should be adapted to the chosen journal's reference style, length limits, and supplementary-data policy.

10. Conclusion and future work

This paper gives a complete and auditable comparison of secure, secure total, and secure connected domination on all thirty connected non-isomorphic graphs of orders two through five. Every graph is drawn and labeled; its edge set and degree sequence are stated; representative minimum sets and exact parameter values are tabulated; and defender certificates are supplied. Two independent implementations reproduce all ninety values.

The finite classification is governed by three patterns. Complete graphs satisfy $(\gamma_s, \gamma_{st}, \gamma_{sc}) = (1,2,1)$. Four dense noncomplete graphs with at least two universal vertices satisfy $(2,2,2)$. The remaining twenty-two graphs satisfy $\gamma_s < \gamma_{st} = \gamma_{sc}$. The general results on noncomplete graphs, universal vertices, complete graphs, and stars provide structural explanations for these observations.

The immediate next step is a complete order-six classification, with special attention to the first possible separation between γ_{st} and γ_{sc} and to degree sequences that support different triples. Further work may seek characterizations for trees, unicyclic graphs, block graphs, and graphs with prescribed numbers of leaves or universal vertices. The certificate format developed here can be retained as the corpus grows, preserving the distinction between computed values and human-checkable mathematical evidence.

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Appendix A. Complete defender certificates

This appendix gives a compact verification record for every graph. For each variant, one minimum witness is stated and one valid defender is selected for each outside vertex.

A.1 Graph $G_{2,1}$

The graph has order $n = 2$, size $m = 1$, degree sequence (1,1), and edge set $E(G_{2,1}) = \{12\}$. Its parameter triple is

$$\left(\gamma_s(G_{2,1}), \gamma_{st}(G_{2,1}), \gamma_{sc}(G_{2,1})\right) = (1,2,1).$$

SDS. A representative minimum set is $S = \{1\}$, so $\gamma_s(G_{2,1}) = 1$. There are 2 minimum SDS witness sets under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2\}$.

STDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_{st}(G_{2,1}) = 2$. There are 1 minimum STDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.

SCDS. A representative minimum set is $S = \{1\}$, so $\gamma_{sc}(G_{2,1}) = 1$. There are 2 minimum SCDS witness sets under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{2,1})$ in increasing order of cardinality. No valid SDS occurs below size 1, no valid STDS occurs below size 2, and no valid SCDS occurs below size 1. The result was reproduced by the independent bitset implementation.

A.2 Graph $G_{3,1}$

The graph has order $n = 3$, size $m = 2$, degree sequence (2,1,1), and edge set $E(G_{3,1}) = \{12,13\}$. Its parameter triple is

$$\left(\gamma_s(G_{3,1}), \gamma_{st}(G_{3,1}), \gamma_{sc}(G_{3,1})\right) = (2,3,3).$$

SDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_s(G_{3,1}) = 2$. There are 3 minimum SDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 1$, giving $\{2,3\}$.

STDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_{st}(G_{3,1}) = 3$. There are 1 minimum STDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.

SCDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_{sc}(G_{3,1}) = 3$. There are 1 minimum SCDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.

Minimality audit. The exhaustive search tested every subset of $V(G_{3,1})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 3, and no valid SCDS occurs below size 3. The result was reproduced by the independent bitset implementation.

A.3 Graph $G_{3,2}$

The graph has order $n = 3$, size $m = 3$, degree sequence (2,2,2), and edge set $E(G_{3,2}) = \{12,13,23\}$. Its parameter triple is

$$\left(\gamma_s(G_{3,2}), \gamma_{st}(G_{3,2}), \gamma_{sc}(G_{3,2})\right) = (1,2,1).$$

SDS. A representative minimum set is $S = \{1\}$, so $\gamma_s(G_{3,2}) = 1$. There are 3 minimum SDS witness sets under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2\}$; $u = 3$ is defended by $v = 1$, giving $\{3\}$.

STDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_{st}(G_{3,2}) = 2$. There are 3 minimum STDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 1$, giving $\{2,3\}$.

SCDS. A representative minimum set is $S = \{1\}$, so $\gamma_{sc}(G_{3,2}) = 1$. There are 3 minimum SCDS witness sets under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2\}$; $u = 3$ is defended by $v = 1$, giving $\{3\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{3,2})$ in increasing order of cardinality. No valid SDS occurs below size 1, no valid STDS occurs below size 2, and no valid SCDS occurs below size 1. The result was reproduced by the independent bitset implementation.

A.4 Graph $G_{4,1}$

The graph has order $n = 4$, size $m = 3$, degree sequence (3,1,1,1), and edge set $E(G_{4,1}) = \{14,24,34\}$. Its parameter triple is

$$\left(\gamma_s(G_{4,1}), \gamma_{st}(G_{4,1}), \gamma_{sc}(G_{4,1})\right) = (3,4,4).$$

SDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_s(G_{4,1}) = 3$. There are 4 minimum SDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 1$, giving $\{2,3,4\}$.
STDS. A representative minimum set is $S = \{1,2,3,4\}$, so $\gamma_{st}(G_{4,1}) = 4$. There are 1 minimum STDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.
SCDS. A representative minimum set is $S = \{1,2,3,4\}$, so $\gamma_{sc}(G_{4,1}) = 4$. There are 1 minimum SCDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.
Minimality audit. The exhaustive search tested every subset of $V(G_{4,1})$ in increasing order of cardinality. No valid SDS occurs below size 3, no valid STDS occurs below size 4, and no valid SCDS occurs below size 4. The result was reproduced by the independent bitset implementation.

A.5 Graph $G_{4,2}$

The graph has order $n = 4$, size $m = 3$, degree sequence (2,2,1,1), and edge set $E(G_{4,2}) = \{12,14,23\}$. Its parameter triple is

$$\left(\gamma_s(G_{4,2}), \gamma_{st}(G_{4,2}), \gamma_{sc}(G_{4,2})\right) = (2,4,4).$$

SDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_s(G_{4,2}) = 2$. There are 4 minimum SDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3\}$; $u = 4$ is defended by $v = 1$, giving $\{2,4\}$.
STDS. A representative minimum set is $S = \{1,2,3,4\}$, so $\gamma_{st}(G_{4,2}) = 4$. There are 1 minimum STDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.
SCDS. A representative minimum set is $S = \{1,2,3,4\}$, so $\gamma_{sc}(G_{4,2}) = 4$. There are 1 minimum SCDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.
Minimality audit. The exhaustive search tested every subset of $V(G_{4,2})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 4, and no valid SCDS occurs below size 4. The result was reproduced by the independent bitset implementation.

A.6 Graph $G_{4,3}$

The graph has order $n = 4$, size $m = 4$, degree sequence (3,2,2,1), and edge set $E(G_{4,3}) = \{14,23,24,34\}$. Its parameter triple is

$$\left(\gamma_s(G_{4,3}), \gamma_{st}(G_{4,3}), \gamma_{sc}(G_{4,3})\right) = (2,3,3).$$

SDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_s(G_{4,3}) = 2$. There are 5 minimum SDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3\}$; $u = 4$ is defended by $v = 1$, giving $\{2,4\}$.
STDS. A representative minimum set is $S = \{1,2,4\}$, so $\gamma_{st}(G_{4,3}) = 3$. There are 2 minimum STDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3,4\}$.
SCDS. A representative minimum set is $S = \{1,2,4\}$, so $\gamma_{sc}(G_{4,3}) = 3$. There are 2 minimum SCDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3,4\}$.
Minimality audit. The exhaustive search tested every subset of $V(G_{4,3})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 3, and no valid SCDS occurs below size 3. The result was reproduced by the independent bitset implementation.

A.7 Graph $G_{4,4}$

The graph has order $n = 4$, size $m = 4$, degree sequence (2,2,2,2), and edge set $E(G_{4,4}) = \{12,14,23,34\}$. Its parameter triple is

$$\left(\gamma_s(G_{4,4}), \gamma_{st}(G_{4,4}), \gamma_{sc}(G_{4,4})\right) = (2,3,3).$$

SDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_s(G_{4,4}) = 2$. There are 6 minimum SDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3\}$; $u = 4$ is defended by $v = 1$, giving $\{2,4\}$.
STDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_{st}(G_{4,4}) = 3$. There are 4 minimum STDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 1$, giving $\{2,3,4\}$.

SCDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_{sc}(G_{4,4}) = 3$. There are 4 minimum SCDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 1$, giving $\{2,3,4\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{4,4})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 3, and no valid SCDS occurs below size 3. The result was reproduced by the independent bitset implementation.

A.8 Graph $G_{4,5}$

The graph has order $n = 4$, size $m = 5$, degree sequence (3,3,2,2), and edge set $E(G_{4,5}) = \{12,13,14,23,34\}$. Its parameter triple is

$$\left(\gamma_s(G_{4,5}), \gamma_{st}(G_{4,5}), \gamma_{sc}(G_{4,5})\right) = (2,2,2).$$

SDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_s(G_{4,5}) = 2$. There are 6 minimum SDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 1$, giving $\{2,3\}$; $u = 4$ is defended by $v = 1$, giving $\{2,4\}$.

STDS. A representative minimum set is $S = \{1,3\}$, so $\gamma_{st}(G_{4,5}) = 2$. There are 1 minimum STDS witness set under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2,3\}$; $u = 4$ is defended by $v = 1$, giving $\{3,4\}$.

SCDS. A representative minimum set is $S = \{1,3\}$, so $\gamma_{sc}(G_{4,5}) = 2$. There are 1 minimum SCDS witness set under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2,3\}$; $u = 4$ is defended by $v = 1$, giving $\{3,4\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{4,5})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 2, and no valid SCDS occurs below size 2. The result was reproduced by the independent bitset implementation.

A.9 Graph $G_{4,6}$

The graph has order $n = 4$, size $m = 6$, degree sequence (3,3,3,3), and edge set $E(G_{4,6}) = \{12,13,14,23,24,34\}$. Its parameter triple is

$$\left(\gamma_s(G_{4,6}), \gamma_{st}(G_{4,6}), \gamma_{sc}(G_{4,6})\right) = (1,2,1).$$

SDS. A representative minimum set is $S = \{1\}$, so $\gamma_s(G_{4,6}) = 1$. There are 4 minimum SDS witness sets under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2\}$; $u = 3$ is defended by $v = 1$, giving $\{3\}$; $u = 4$ is defended by $v = 1$, giving $\{4\}$.

STDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_{st}(G_{4,6}) = 2$. There are 6 minimum STDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 1$, giving $\{2,3\}$; $u = 4$ is defended by $v = 1$, giving $\{2,4\}$.

SCDS. A representative minimum set is $S = \{1\}$, so $\gamma_{sc}(G_{4,6}) = 1$. There are 4 minimum SCDS witness sets under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2\}$; $u = 3$ is defended by $v = 1$, giving $\{3\}$; $u = 4$ is defended by $v = 1$, giving $\{4\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{4,6})$ in increasing order of cardinality. No valid SDS occurs below size 1, no valid STDS occurs below size 2, and no valid SCDS occurs below size 1. The result was reproduced by the independent bitset implementation.

A.10 Graph $G_{5,1}$

The graph has order $n = 5$, size $m = 4$, degree sequence (4,1,1,1,1), and edge set $E(G_{5,1}) = \{15,25,35,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,1}), \gamma_{st}(G_{5,1}), \gamma_{sc}(G_{5,1})\right) = (4,5,5).$$

SDS. A representative minimum set is $S = \{1,2,3,4\}$, so $\gamma_s(G_{5,1}) = 4$. There are 5 minimum SDS witness sets under the displayed labeling. $u = 5$ is defended by $v = 1$, giving $\{2,3,4,5\}$.

STDS. A representative minimum set is $S = \{1,2,3,4,5\}$, so $\gamma_{st}(G_{5,1}) = 5$. There are 1 minimum STDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.

SCDS. A representative minimum set is $S = \{1,2,3,4,5\}$, so $\gamma_{sc}(G_{5,1}) = 5$. There are 1 minimum SCDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,1})$ in increasing order of cardinality. No valid SDS occurs below size 4, no valid STDS occurs below size 5, and no valid SCDS occurs below size 5. The result was reproduced by the independent bitset implementation.

A.11 Graph $G_{5,2}$

The graph has order $n = 5$, size $m = 4$, degree sequence (3,2,1,1,1), and edge set $E(G_{5,2}) = \{15,24,34,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,2}), \gamma_{st}(G_{5,2}), \gamma_{sc}(G_{5,2})\right) = (3,5,5).$$

SDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_s(G_{5,2}) = 3$. There are 6 minimum SDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 2$, giving $\{1,3,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,3,5\}$.

STDS. A representative minimum set is $S = \{1,2,3,4,5\}$, so $\gamma_{st}(G_{5,2}) = 5$. There are 1 minimum STDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.

SCDS. A representative minimum set is $S = \{1,2,3,4,5\}$, so $\gamma_{sc}(G_{5,2}) = 5$. There are 1 minimum SCDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,2})$ in increasing order of cardinality. No valid SDS occurs below size 3, no valid STDS occurs below size 5, and no valid SCDS occurs below size 5. The result was reproduced by the independent bitset implementation.

A.12 Graph $G_{5,3}$

The graph has order $n = 5$, size $m = 4$, degree sequence (2,2,2,1,1), and edge set $E(G_{5,3}) = \{12,15,23,34\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,3}), \gamma_{st}(G_{5,3}), \gamma_{sc}(G_{5,3})\right) = (3,5,5).$$

SDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_s(G_{5,3}) = 3$. There are 8 minimum SDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 3$, giving $\{1,2,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,3,5\}$.

STDS. A representative minimum set is $S = \{1,2,3,4,5\}$, so $\gamma_{st}(G_{5,3}) = 5$. There are 1 minimum STDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.

SCDS. A representative minimum set is $S = \{1,2,3,4,5\}$, so $\gamma_{sc}(G_{5,3}) = 5$. There are 1 minimum SCDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,3})$ in increasing order of cardinality. No valid SDS occurs below size 3, no valid STDS occurs below size 5, and no valid SCDS occurs below size 5. The result was reproduced by the independent bitset implementation.

A.13 Graph $G_{5,4}$

The graph has order $n = 5$, size $m = 5$, degree sequence (4,2,2,1,1), and edge set $E(G_{5,4}) = \{15,25,34,35,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,4}), \gamma_{st}(G_{5,4}), \gamma_{sc}(G_{5,4})\right) = (3,4,4).$$

SDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_s(G_{5,4}) = 3$. There are 7 minimum SDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 3$, giving $\{1,2,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,3,5\}$.

STDS. A representative minimum set is $S = \{1,2,3,5\}$, so $\gamma_{st}(G_{5,4}) = 4$. There are 2 minimum STDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 3$, giving $\{1,2,4,5\}$.

SCDS. A representative minimum set is $S = \{1,2,3,5\}$, so $\gamma_{sc}(G_{5,4}) = 4$. There are 2 minimum SCDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 3$, giving $\{1,2,4,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,4})$ in increasing order of cardinality. No valid SDS occurs below size 3, no valid STDS occurs below size 4, and no valid SCDS occurs below size 4. The result was reproduced by the independent bitset implementation.

A.14 Graph $G_{5,5}$

The graph has order $n = 5$, size $m = 5$, degree sequence (3,3,2,1,1), and edge set $E(G_{5,5}) = \{12,13,15,23,34\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,5}), \gamma_{st}(G_{5,5}), \gamma_{sc}(G_{5,5})\right) = (3,5,5).$$

SDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_s(G_{5,5}) = 3$. There are 8 minimum SDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 3$, giving $\{1,2,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,3,5\}$.

STDS. A representative minimum set is $S = \{1,2,3,4,5\}$, so $\gamma_{st}(G_{5,5}) = 5$. There are 1 minimum STDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.

SCDS. A representative minimum set is $S = \{1,2,3,4,5\}$, so $\gamma_{sc}(G_{5,5}) = 5$. There are 1 minimum SCDS witness set under the displayed labeling. No outside vertex; the exchange condition is vacuous.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,5})$ in increasing order of cardinality. No valid SDS occurs below size 3, no valid STDS occurs below size 5, and no valid SCDS occurs below size 5. The result was reproduced by the independent bitset implementation.

A.15 Graph $G_{5,6}$

The graph has order $n = 5$, size $m = 5$, degree sequence (3,2,2,2,1), and edge set $E(G_{5,6}) = \{15,23,24,34,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,6}), \gamma_{st}(G_{5,6}), \gamma_{sc}(G_{5,6})\right) = (2,4,4).$$

SDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_s(G_{5,6}) = 2$. There are 6 minimum SDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3\}$; $u = 4$ is defended by $v = 2$, giving $\{1,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,5\}$.

STDS. A representative minimum set is $S = \{1,2,3,5\}$, so $\gamma_{st}(G_{5,6}) = 4$. There are 3 minimum STDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 2$, giving $\{1,3,4,5\}$.

SCDS. A representative minimum set is $S = \{1,2,4,5\}$, so $\gamma_{sc}(G_{5,6}) = 4$. There are 2 minimum SCDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3,4,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,6})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 4, and no valid SCDS occurs below size 4. The result was reproduced by the independent bitset implementation.

A.16 Graph $G_{5,7}$

The graph has order $n = 5$, size $m = 5$, degree sequence (3,2,2,2,1), and edge set $E(G_{5,7}) = \{12,24,25,34,35\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,7}), \gamma_{st}(G_{5,7}), \gamma_{sc}(G_{5,7})\right) = (2,4,4).$$

SDS. A representative minimum set is $S = \{2,3\}$, so $\gamma_s(G_{5,7}) = 2$. There are 1 minimum SDS witness set under the displayed labeling. $u = 1$ is defended by $v = 2$, giving $\{1,3\}$; $u = 4$ is defended by $v = 3$, giving $\{2,4\}$; $u = 5$ is defended by $v = 3$, giving $\{2,5\}$.

STDS. A representative minimum set is $S = \{1,2,3,4\}$, so $\gamma_{st}(G_{5,7}) = 4$. There are 3 minimum STDS witness sets under the displayed labeling. $u = 5$ is defended by $v = 3$, giving $\{1,2,4,5\}$.

SCDS. A representative minimum set is $S = \{1,2,3,4\}$, so $\gamma_{sc}(G_{5,7}) = 4$. There are 3 minimum SCDS witness sets under the displayed labeling. $u = 5$ is defended by $v = 3$, giving $\{1,2,4,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,7})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 4, and no valid SCDS occurs below size 4. The result was reproduced by the independent bitset implementation.

A.17 Graph $G_{5,8}$

The graph has order $n = 5$, size $m = 5$, degree sequence (2,2,2,2,2), and edge set $E(G_{5,8}) = \{12,15,23,34,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,8}), \gamma_{st}(G_{5,8}), \gamma_{sc}(G_{5,8})\right) = (3,4,4).$$

SDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_s(G_{5,8}) = 3$. There are 10 minimum SDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 3$, giving $\{1,2,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,3,5\}$.

STDS. A representative minimum set is $S = \{1,2,3,4\}$, so $\gamma_{st}(G_{5,8}) = 4$. There are 5 minimum STDS witness sets under the displayed labeling. $u = 5$ is defended by $v = 1$, giving $\{2,3,4,5\}$.

SCDS. A representative minimum set is $S = \{1,2,3,4\}$, so $\gamma_{sc}(G_{5,8}) = 4$. There are 5 minimum SCDS witness sets under the displayed labeling. $u = 5$ is defended by $v = 1$, giving $\{2,3,4,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,8})$ in increasing order of cardinality. No valid SDS occurs below size 3, no valid STDS occurs below size 4, and no valid SCDS occurs below size 4. The result was reproduced by the independent bitset implementation.

A.18 Graph $G_{5,9}$

The graph has order $n = 5$, size $m = 6$, degree sequence (4,3,2,2,1), and edge set $E(G_{5,9}) = \{12,23,24,25,34,35\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,9}), \gamma_{st}(G_{5,9}), \gamma_{sc}(G_{5,9})\right) = (2,3,3).$$

SDS. A representative minimum set is $S = \{2,3\}$, so $\gamma_s(G_{5,9}) = 2$. There are 1 minimum SDS witness set under the displayed labeling. $u = 1$ is defended by $v = 2$, giving $\{1,3\}$; $u = 4$ is defended by $v = 3$, giving $\{2,4\}$; $u = 5$ is defended by $v = 3$, giving $\{2,5\}$.

STDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_{st}(G_{5,9}) = 3$. There are 1 minimum STDS witness set under the displayed labeling. $u = 4$ is defended by $v = 3$, giving $\{1,2,4\}$; $u = 5$ is defended by $v = 3$, giving $\{1,2,5\}$.

SCDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_{sc}(G_{5,9}) = 3$. There are 1 minimum SCDS witness set under the displayed labeling. $u = 4$ is defended by $v = 3$, giving $\{1,2,4\}$; $u = 5$ is defended by $v = 3$, giving $\{1,2,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,9})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 3, and no valid SCDS occurs below size 3. The result was reproduced by the independent bitset implementation.

A.19 Graph $G_{5,10}$

The graph has order $n = 5$, size $m = 6$, degree sequence (3,3,3,2,1), and edge set $E(G_{5,10}) = \{12,24,25,34,35,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,10}), \gamma_{st}(G_{5,10}), \gamma_{sc}(G_{5,10})\right) = (2,4,4).$$

SDS. A representative minimum set is $S = \{1,3\}$, so $\gamma_s(G_{5,10}) = 2$. There are 6 minimum SDS witness sets under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2,3\}$; $u = 4$ is defended by $v = 3$, giving $\{1,4\}$; $u = 5$ is defended by $v = 3$, giving $\{1,5\}$.

STDS. A representative minimum set is $S = \{1,2,3,4\}$, so $\gamma_{st}(G_{5,10}) = 4$. There are 3 minimum STDS witness sets under the displayed labeling. $u = 5$ is defended by $v = 3$, giving $\{1,2,4,5\}$.

SCDS. A representative minimum set is $S = \{1,2,3,4\}$, so $\gamma_{sc}(G_{5,10}) = 4$. There are 3 minimum SCDS witness sets under the displayed labeling. $u = 5$ is defended by $v = 3$, giving $\{1,2,4,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,10})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 4, and no valid SCDS occurs below size 4. The result was reproduced by the independent bitset implementation.

A.20 Graph $G_{5,11}$

The graph has order $n = 5$, size $m = 6$, degree sequence (4,2,2,2,2), and edge set $E(G_{5,11}) = \{12,15,25,34,35,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,11}), \gamma_{st}(G_{5,11}), \gamma_{sc}(G_{5,11})\right) = (2,3,3).$$

SDS. A representative minimum set is $S = \{1,3\}$, so $\gamma_s(G_{5,11}) = 2$. There are 8 minimum SDS witness sets under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2,3\}$; $u = 4$ is defended by $v = 3$, giving $\{1,4\}$; $u = 5$ is defended by $v = 1$, giving $\{3,5\}$.

STDS. A representative minimum set is $S = \{1,3,5\}$, so $\gamma_{st}(G_{5,11}) = 3$. There are 4 minimum STDS witness sets under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2,3,5\}$; $u = 4$ is defended by $v = 3$, giving $\{1,4,5\}$.

SCDS. A representative minimum set is $S = \{1,3,5\}$, so $\gamma_{sc}(G_{5,11}) = 3$. There are 4 minimum SCDS witness sets under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2,3,5\}$; $u = 4$ is defended by $v = 3$, giving $\{1,4,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,11})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 3, and no valid SCDS occurs below size 3. The result was reproduced by the independent bitset implementation.

A.21 Graph $G_{5,12}$

The graph has order $n = 5$, size $m = 6$, degree sequence (3,3,2,2,2), and edge set $E(G_{5,12}) = \{12,14,15,23,34,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,12}), \gamma_{st}(G_{5,12}), \gamma_{sc}(G_{5,12})\right) = (2,3,3).$$

SDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_s(G_{5,12}) = 2$. There are 6 minimum SDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3\}$; $u = 4$ is defended by $v = 1$, giving $\{2,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,5\}$.

STDS. A representative minimum set is $S = \{1,2,4\}$, so $\gamma_{st}(G_{5,12}) = 3$. There are 2 minimum STDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3,4\}$; $u = 5$ is defended by $v = 4$, giving $\{1,2,5\}$.

SCDS. A representative minimum set is $S = \{1,2,4\}$, so $\gamma_{sc}(G_{5,12}) = 3$. There are 2 minimum SCDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3,4\}$; $u = 5$ is defended by $v = 4$, giving $\{1,2,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,12})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 3, and no valid SCDS occurs below size 3. The result was reproduced by the independent bitset implementation.

A.22 Graph $G_{5,13}$

The graph has order $n = 5$, size $m = 6$, degree sequence (3,3,2,2,2), and edge set $E(G_{5,13}) = \{13,14,15,23,24,25\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,13}), \gamma_{st}(G_{5,13}), \gamma_{sc}(G_{5,13})\right) = (2,3,3).$$

SDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_s(G_{5,13}) = 2$. There are 1 minimum SDS witness set under the displayed labeling. $u = 3$ is defended by $v = 1$, giving $\{2,3\}$; $u = 4$ is defended by $v = 1$, giving $\{2,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,5\}$.

STDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_{st}(G_{5,13}) = 3$. There are 3 minimum STDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 1$, giving $\{2,3,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,3,5\}$.

SCDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_{sc}(G_{5,13}) = 3$. There are 3 minimum SCDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 1$, giving $\{2,3,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,3,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,13})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 3, and no valid SCDS occurs below size 3. The result was reproduced by the independent bitset implementation.

A.23 Graph $G_{5,14}$

The graph has order $n = 5$, size $m = 7$, degree sequence (4,3,3,3,1), and edge set $E(G_{5,14}) = \{15,23,24,25,34,35,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,14}), \gamma_{st}(G_{5,14}), \gamma_{sc}(G_{5,14})\right) = (2,3,3).$$

SDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_s(G_{5,14}) = 2$. There are 7 minimum SDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3\}$; $u = 4$ is defended by $v = 2$, giving $\{1,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,5\}$.

STDS. A representative minimum set is $S = \{1,2,5\}$, so $\gamma_{st}(G_{5,14}) = 3$. There are 3 minimum STDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3,5\}$; $u = 4$ is defended by $v = 2$, giving $\{1,4,5\}$.

SCDS. A representative minimum set is $S = \{1,2,5\}$, so $\gamma_{sc}(G_{5,14}) = 3$. There are 3 minimum SCDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3,5\}$; $u = 4$ is defended by $v = 2$, giving $\{1,4,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,14})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 3, and no valid SCDS occurs below size 3. The result was reproduced by the independent bitset implementation.

A.24 Graph $G_{5,15}$

The graph has order $n = 5$, size $m = 7$, degree sequence (4,4,2,2,2), and edge set $E(G_{5,15}) = \{14,15,24,25,34,35,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,15}), \gamma_{st}(G_{5,15}), \gamma_{sc}(G_{5,15})\right) = (2,2,2).$$

SDS. A representative minimum set is $S = \{4,5\}$, so $\gamma_s(G_{5,15}) = 2$. There are 1 minimum SDS witness set under the displayed labeling. $u = 1$ is defended by $v = 4$, giving $\{1,5\}$; $u = 2$ is defended by $v = 4$, giving $\{2,5\}$; $u = 3$ is defended by $v = 4$, giving $\{3,5\}$.

STDS. A representative minimum set is $S = \{4,5\}$, so $\gamma_{st}(G_{5,15}) = 2$. There are 1 minimum STDS witness set under the displayed labeling. $u = 1$ is defended by $v = 4$, giving $\{1,5\}$; $u = 2$ is defended by $v = 4$, giving $\{2,5\}$; $u = 3$ is defended by $v = 4$, giving $\{3,5\}$.

SCDS. A representative minimum set is $S = \{4,5\}$, so $\gamma_{sc}(G_{5,15}) = 2$. There are 1 minimum SCDS witness set under the displayed labeling. $u = 1$ is defended by $v = 4$, giving $\{1,5\}$; $u = 2$ is defended by $v = 4$, giving $\{2,5\}$; $u = 3$ is defended by $v = 4$, giving $\{3,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,15})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 2, and no valid SCDS occurs below size 2. The result was reproduced by the independent bitset implementation.

A.25 Graph $G_{5,16}$

The graph has order $n = 5$, size $m = 7$, degree sequence (4,3,3,2,2), and edge set $E(G_{5,16}) = \{12,15,23,24,25,34,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,16}), \gamma_{st}(G_{5,16}), \gamma_{sc}(G_{5,16})\right) = (2,3,3).$$

SDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_s(G_{5,16}) = 2$. There are 8 minimum SDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3\}$; $u = 4$ is defended by $v = 2$, giving $\{1,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,5\}$.

STDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_{st}(G_{5,16}) = 3$. There are 4 minimum STDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 3$, giving $\{1,2,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,3,5\}$.

SCDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_{sc}(G_{5,16}) = 3$. There are 4 minimum SCDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 3$, giving $\{1,2,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,3,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,16})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 3, and no valid SCDS occurs below size 3. The result was reproduced by the independent bitset implementation.

A.26 Graph $G_{5,17}$

The graph has order $n = 5$, size $m = 7$, degree sequence (3,3,3,3,2), and edge set $E(G_{5,17}) = \{13,14,15,23,24,25,35\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,17}), \gamma_{st}(G_{5,17}), \gamma_{sc}(G_{5,17})\right) = (2,3,3).$$

SDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_s(G_{5,17}) = 2$. There are 9 minimum SDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 1$, giving $\{2,3\}$; $u = 4$ is defended by $v = 1$, giving $\{2,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,5\}$.

STDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_{st}(G_{5,17}) = 3$. There are 7 minimum STDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 1$, giving $\{2,3,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,3,5\}$.

SCDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_{sc}(G_{5,17}) = 3$. There are 7 minimum SCDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 1$, giving $\{2,3,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,3,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,17})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 3, and no valid SCDS occurs below size 3. The result was reproduced by the independent bitset implementation.

A.27 Graph $G_{5,18}$

The graph has order $n = 5$, size $m = 8$, degree sequence (4,4,3,3,2), and edge set $E(G_{5,18}) = \{12,14,15,24,25,34,35,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,18}), \gamma_{st}(G_{5,18}), \gamma_{sc}(G_{5,18})\right) = (2,2,2).$$

SDS. A representative minimum set is $S = \{1,3\}$, so $\gamma_s(G_{5,18}) = 2$. There are 9 minimum SDS witness sets under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2,3\}$; $u = 4$ is defended by $v = 1$, giving $\{3,4\}$; $u = 5$ is defended by $v = 1$, giving $\{3,5\}$.

STDS. A representative minimum set is $S = \{4,5\}$, so $\gamma_{st}(G_{5,18}) = 2$. There are 1 minimum STDS witness set under the displayed labeling. $u = 1$ is defended by $v = 4$, giving $\{1,5\}$; $u = 2$ is defended by $v = 4$, giving $\{2,5\}$; $u = 3$ is defended by $v = 4$, giving $\{3,5\}$.

SCDS. A representative minimum set is $S = \{4,5\}$, so $\gamma_{sc}(G_{5,18}) = 2$. There are 1 minimum SCDS witness set under the displayed labeling. $u = 1$ is defended by $v = 4$, giving $\{1,5\}$; $u = 2$ is defended by $v = 4$, giving $\{2,5\}$; $u = 3$ is defended by $v = 4$, giving $\{3,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,18})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 2, and no valid SCDS occurs below size 2. The result was reproduced by the independent bitset implementation.

A.28 Graph $G_{5,19}$

The graph has order $n = 5$, size $m = 8$, degree sequence (4,3,3,3,3), and edge set $E(G_{5,19}) = \{12,14,15,23,25,34,35,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,19}), \gamma_{st}(G_{5,19}), \gamma_{sc}(G_{5,19})\right) = (2,3,3).$$

SDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_s(G_{5,19}) = 2$. There are 10 minimum SDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3\}$; $u = 4$ is defended by $v = 1$, giving $\{2,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,5\}$.

STDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_{st}(G_{5,19}) = 3$. There are 10 minimum STDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 1$, giving $\{2,3,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,3,5\}$.

SCDS. A representative minimum set is $S = \{1,2,3\}$, so $\gamma_{sc}(G_{5,19}) = 3$. There are 10 minimum SCDS witness sets under the displayed labeling. $u = 4$ is defended by $v = 1$, giving $\{2,3,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,3,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,19})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 3, and no valid SCDS occurs below size 3. The result was reproduced by the independent bitset implementation.

A.29 Graph $G_{5,20}$

The graph has order $n = 5$, size $m = 9$, degree sequence (4,4,4,3,3), and edge set $E(G_{5,20}) = \{12,14,15,23,24,25,34,35,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,20}), \gamma_{st}(G_{5,20}), \gamma_{sc}(G_{5,20})\right) = (2,2,2).$$

SDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_s(G_{5,20}) = 2$. There are 10 minimum SDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 2$, giving $\{1,3\}$; $u = 4$ is defended by $v = 1$, giving $\{2,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,5\}$.

STDS. A representative minimum set is $S = \{2,4\}$, so $\gamma_{st}(G_{5,20}) = 2$. There are 3 minimum STDS witness sets under the displayed labeling. $u = 1$ is defended by $v = 2$, giving $\{1,4\}$; $u = 3$ is defended by $v = 2$, giving $\{3,4\}$; $u = 5$ is defended by $v = 2$, giving $\{4,5\}$.

SCDS. A representative minimum set is $S = \{2,4\}$, so $\gamma_{sc}(G_{5,20}) = 2$. There are 3 minimum SCDS witness sets under the displayed labeling. $u = 1$ is defended by $v = 2$, giving $\{1,4\}$; $u = 3$ is defended by $v = 2$, giving $\{3,4\}$; $u = 5$ is defended by $v = 2$, giving $\{4,5\}$.

Minimality audit. The exhaustive search tested every subset of $V(G_{5,20})$ in increasing order of cardinality. No valid SDS occurs below size 2, no valid STDS occurs below size 2, and no valid SCDS occurs below size 2. The result was reproduced by the independent bitset implementation.

A.30 Graph $G_{5,21}$

The graph has order $n = 5$, size $m = 10$, degree sequence (4,4,4,4,4), and edge set $E(G_{5,21}) = \{12,13,14,15,23,24,25,34,35,45\}$. Its parameter triple is

$$\left(\gamma_s(G_{5,21}), \gamma_{st}(G_{5,21}), \gamma_{sc}(G_{5,21})\right) = (1,2,1).$$

SDS. A representative minimum set is $S = \{1\}$, so $\gamma_s(G_{5,21}) = 1$. There are 5 minimum SDS witness sets under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2\}$; $u = 3$ is defended by $v = 1$, giving $\{3\}$; $u = 4$ is defended by $v = 1$, giving $\{4\}$; $u = 5$ is defended by $v = 1$, giving $\{5\}$.
STDS. A representative minimum set is $S = \{1,2\}$, so $\gamma_{st}(G_{5,21}) = 2$. There are 10 minimum STDS witness sets under the displayed labeling. $u = 3$ is defended by $v = 1$, giving $\{2,3\}$; $u = 4$ is defended by $v = 1$, giving $\{2,4\}$; $u = 5$ is defended by $v = 1$, giving $\{2,5\}$.
SCDS. A representative minimum set is $S = \{1\}$, so $\gamma_{sc}(G_{5,21}) = 1$. There are 5 minimum SCDS witness sets under the displayed labeling. $u = 2$ is defended by $v = 1$, giving $\{2\}$; $u = 3$ is defended by $v = 1$, giving $\{3\}$; $u = 4$ is defended by $v = 1$, giving $\{4\}$; $u = 5$ is defended by $v = 1$, giving $\{5\}$.
Minimality audit. The exhaustive search tested every subset of $V(G_{5,21})$ in increasing order of cardinality. No valid SDS occurs below size 1, no valid STDS occurs below size 2, and no valid SCDS occurs below size 1. The result was reproduced by the independent bitset implementation.

Appendix B. Complete edge-list index

The following tables provide a compact textual reconstruction of every graph, independent of the drawings.

Table 13. Edge-list index for order **2**.

Graph	n	m	Degree sequence	Edge set	Universal vertices
$G_{2,1}$	2	1	(1,1)	{12}	{1,2}

Table 14. Edge-list index for order **3**.

Graph	n	m	Degree sequence	Edge set	Universal vertices
$G_{3,1}$	3	2	(2,1,1)	{12,13}	{1}
$G_{3,2}$	3	3	(2,2,2)	{12,13,23}	{1,2,3}

Table 15. Edge-list index for order **4**.

Graph	n	m	Degree sequence	Edge set	Universal vertices
$G_{4,1}$	4	3	(3,1,1,1)	{14,24,34}	{4}
$G_{4,2}$	4	3	(2,2,1,1)	{12,14,23}	⌀
$G_{4,3}$	4	4	(3,2,2,1)	{14,23,24,34}	{4}
$G_{4,4}$	4	4	(2,2,2,2)	{12,14,23,34}	⌀
$G_{4,5}$	4	5	(3,3,2,2)	{12,13,14,23,34}	{1,3}
$G_{4,6}$	4	6	(3,3,3,3)	{12,13,14,23,24,34}	{1,2,3,4}

Table 16. Edge-list index for order **5**.

Graph	n	m	Degree sequence	Edge set	Universal vertices
$G_{5,1}$	5	4	(4,1,1,1,1)	{15,25,35,45}	{5}
$G_{5,2}$	5	4	(3,2,1,1,1)	{15,24,34,45}	⌀

Graph	n	m	Degree sequence	Edge set	Universal vertices
$G_{5,3}$	5	4	(2,2,2,1,1)	{12,15,23,34}	⌀
$G_{5,4}$	5	5	(4,2,2,1,1)	{15,25,34,35,45}	{5}
$G_{5,5}$	5	5	(3,3,2,1,1)	{12,13,15,23,34}	⌀
$G_{5,6}$	5	5	(3,2,2,2,1)	{15,23,24,34,45}	⌀
$G_{5,7}$	5	5	(3,2,2,2,1)	{12,24,25,34,35}	⌀
$G_{5,8}$	5	5	(2,2,2,2,2)	{12,15,23,34,45}	⌀
$G_{5,9}$	5	6	(4,3,2,2,1)	{12,23,24,25,34,35}	{2}
$G_{5,10}$	5	6	(3,3,3,2,1)	{12,24,25,34,35,45}	⌀
$G_{5,11}$	5	6	(4,2,2,2,2)	{12,15,25,34,35,45}	{5}
$G_{5,12}$	5	6	(3,3,2,2,2)	{12,14,15,23,34,45}	⌀
$G_{5,13}$	5	6	(3,3,2,2,2)	{13,14,15,23,24,25}	⌀
$G_{5,14}$	5	7	(4,3,3,3,1)	{15,23,24,25,34,35,45}	{5}
$G_{5,15}$	5	7	(4,4,2,2,2)	{14,15,24,25,34,35,45}	{4,5}
$G_{5,16}$	5	7	(4,3,3,2,2)	{12,15,23,24,25,34,45}	{2}
$G_{5,17}$	5	7	(3,3,3,3,2)	{13,14,15,23,24,25,35}	⌀
$G_{5,18}$	5	8	(4,4,3,3,2)	{12,14,15,24,25,34,35,45}	{4,5}
$G_{5,19}$	5	8	(4,3,3,3,3)	{12,14,15,23,25,34,35,45}	{5}
$G_{5,20}$	5	9	(4,4,4,3,3)	{12,14,15,23,24,25,34,35,45}	{2,4,5}
$G_{5,21}$	5	10	(4,4,4,4,4)	{12,13,14,15,23,24,25,34,35,45}	{1,2,3,4,5}