

ON S-PURE SUBMODULES

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Abstract: This paper explores various characterizations and properties of S -pure submodules, including their relationships with other types of submodules such as extended S -pure submodules and extended S -flat modules. We investigate conditions under which extended S -pure submodules are preserved under certain operations, such as taking direct sums and extensions. Additionally, we examine examples illustrating the significance of extended S -purity in different algebraic contexts, including its implications for projective modules and flat modules. We introduce some concepts class of modules relative to a semiradical property. Let S semiradical property and M be a module. We say that N extended S -pure submodule if there is K submodule of M such that $N \cap K = 0$ and $S(M/(N \oplus K)) = 0$. In the first part, we define a submodule $S_2(M)$ by the equality $S_2(M)/(S(M) = S(M/S(M))$, for any module M . A module M is called S -Goldie module if $S_2(M) = M$. Also, we introduce the concept extended S -pure submodule. We consider the relations between the second semiradical (S -Goldie) submodule and extended S -pure submodule. In the second part, we introduce the concept of extended S -flat module. M is called es -flat module if for any epimorphism $f: N \rightarrow M$, $\ker f$ is extended S -pure submodule in N . Also, some basic properties are investigated.

Introduction

Extended S -pure submodules are useful in various areas such as module theory and homological algebra. They help in studying class of modules by providing insights into their structure. Understanding extended S -pure submodules can lead to characterizations of certain classes of modules (e.g., es -flat modules), which are important for many theoretical developments and applications. Throughout this paper, R is a ring with identity, M is a until left module. The observes $K \leq M$, $K \leqslant M$, $K \leq_{\oplus} M$, $K \leq_{s,p} M$, $K \leq_{esp} M$ mean that K is a submodule, proper submodule, direct summand, S -pure submodule, extended S -pure submodule of M . In this paper, we mean S is a semiradical property unless otherwise stated. Whereas i is mean the inclusion map and π is mean the natural epimorphism. Recall that K is said to be a pure submodule of M , if $K \cap IM = IK$, for every finitely generated ideal I of R , see [1]. The singular submodule of M define as $Z(M) = \{a \in M : \text{ann}(a) \leq_e R\}$. M is said a singular module if $Z(M) = M$ and M is said to be a nonsingular module if $Z(M) = 0$, see [10]. Recall that M is called a faithful module if $\text{ann}(M) = 0$, where $\text{ann}(M) = \{r \in R \mid rx = 0, \forall x \in M\}$, see [6]. Let M be a module, the Jacobson radical of M , $J(M) = \bigcap_{K \leq M_1} K$. If M has no maximal K is maximal

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submodule, then $J(M) = M$, see [6]. The socle of M , $Soc(M) = \sum_{\substack{N \leq M \\ N \text{ is simple}}} N$, see [4].

M is called a semiartinian module (briefly *Sa*-module), if for every $N \leq M$, $Soc\left(\frac{M}{N}\right) \neq 0$, see [5]. N is said to be *s*-pure submodule of M if and only if $S\left(\frac{M}{N}\right) = 0$.

The Closed submodule form an important role in rings and module theory. Many researchers studies have explored various generalizations of closed submodules. For instance, *S*-closed submodules defined by Goodearl in [8]. N is called *S*-closed submodule of M if $\frac{M}{N}$ is nonsingular. Moreover, *T*-closed submodules defined in [4] *D*-closed submodules defined in [7]. Yilmaz and Salahattin in [3] introduced extended *S*-closed submodules. N is called extended *S*-closed submodule of M if there is $N_1 \leq M$ such that $N \cap N_1 = 0$ and $\frac{M}{N \oplus N_1}$ is nonsingular. N. Hamad and B. AL-Hashmi in [2] introduced the concept of semiradical property, A property *S* is said to be a semiradical property if:

1- For every module M , there is a submodule (briefly $S(M)$) such that:

a- $N \leq S(M)$, for every $N \leq M$ such that $S(N) = N$.

b- $S(S(M)) = S(M)$.

2- If $f: M \rightarrow N$ is epimorphism and $S(M) = M$, then $S(N) = N$.

A semiradical property is said to be radical property if $S\left(\frac{M}{S(M)}\right) = 0$, for each module M .

The following three properties are semiradical property.

1- $S = Z$. For a module M , let $S(M) = Z(M)$ the singular submodule of M . Z is a semiradical property, see [2].

2- $S = Soc$. For a module M , let $S(M) = Soc(M)$ the socle submodule of M . Soc is a semiradical property, see [2].

3- $S = \mathcal{M}$ For a module M , let $S(M) = \mathcal{M}(M) = \sum_{\substack{K \leq M \\ K \text{ is regular}}} K$, $\mathcal{M}(M)$ is called semi Broun-McCoy radical). $\mathcal{M}$ is a semiradical property, see [2].

While the two properties are a radical property.

1- $S = Snr$. For a module M , let $S(M) = Snr(M) = \sum_{\substack{K \leq M \\ J(K)=K}} K$, where $J(K)$ is the Jacobson radical of K . It's known that Snr is a radical property, see [2].

2- $S = Sa$. For a module M , let $S(M) = Sa(M) = \sum_{\substack{A \leq M \\ A \text{ is Sa}}} A$, $Sa(M)$ is called the semiartinian submodule of M . It's known that Sa is a radical property, see [2].

A module M is said to be a flat if and only if every short exact sequence $0 \rightarrow K \rightarrow N \rightarrow M \rightarrow 0$ is pure exact, see This relationship between the concepts of purity and flatness has lead us to further research is needed on some classes of modules defined by means of pure submodules. Also, a module M is called extended *S*-closed flat (or *ec*-flat for short) module if the kernel of any epimorphism $f: N \rightarrow M$, is extended *S*-closed submodule of N , see [3].

These observations lead us to introduce the concepts extended *S*-pure submodule and extended *S*-flat module. Let S semiradical property and $N \leq M$. We can say that N extended *S*-pure (briefly *es*-pure) submodule of M if there is N_1 submodule of M such that $N \cap N_1 = 0$ and $S\left(\frac{M}{N \oplus N_1}\right) = 0$. Moreover, a module M is extended *S*-flat (briefly *es*-flat) module if the kernel of any epimorphism $f: N \rightarrow M$, is *es*-pure in N .

The work comprises three sections. In Section two we introduce the concept of the second semiradical (or S -Goldie) submodule $S_2(M)$ of M . We illustrate some examples and provides some properties. For instance, every module has S is S -Goldie module. The converse is true if S is a radical property. Also, we prove that, $S(N) = 0$, for any module N if and only if $\text{Hom}(S_2(M), N) = 0$, for every module M . In Section three, we introduce the concept extended S -pure submodule. We illustrate some examples and provides some properties are investigated. For instance, let $N \leq M$. If $N \leq_{s.p} M$, then $N \leq_{esp} M$. In addition, if $N \leq_{\oplus} M$, then $N \leq_{esp} M$. We show that, If $N \leq_{esp} M$, then every S -Goldie module K is projective with regard to the short exact sequence $0 \rightarrow N \rightarrow M \xrightarrow{i} \frac{M}{N} \rightarrow 0$. The converse is true if $S\left(\frac{M}{S_2(M)}\right) = 0$, in particular S is a radical property. In Section four, we introduce the concept of extended S -flat module. We illustrate some examples and provides some properties are investigated. For instance, if M be a module with $S(M) = 0$ (projective), then M is es -flat module. In addition, let S be a radical property, M is es -flat module if and only if there is an es -pure sequence $0 \rightarrow K \rightarrow N \rightarrow M \rightarrow 0$ with N is a projective (es -flat). We show that, a direct summand of es -flat is es -flat module. Also, let $0 \rightarrow K \xrightarrow{f} N \xrightarrow{g} M \rightarrow 0$ be a short exact sequence. If K and M are es -flat modules, then N is es -flat module. We show that, A S -Goldie module K is es -flat module if and only if K is a projective. Moreover, every cyclic es -flat module is projective, then $R \cong M \oplus N$, where M is S -Goldie module and N is a projective.

2-The Second Semiradical Submodule

This section introduces the concept of the second semiradical submodule. We illustrate is by some examples. We also give some basic properties.

Definition 2.1

Let M be a module. The second semiradical (S -Goldie) submodule $S_2(M)$ is defined by the equality $\frac{S_2(M)}{S(M)} = S\left(\frac{M}{S(M)}\right)$. M is called S -Goldie module if $S_2(M) = M$.

Remarks and examples 2.2

1- Let $S = Z$ and consider the module Q as Z -module. Since $S(Q) = 0$, then

$$\frac{S_2(Q)}{S(Q)} = S\left(\frac{Q}{S(Q)}\right) = S\left(\frac{Q}{0}\right) = S(Q) = 0. \text{ Thus } S_2(Q) = S(Q) = 0.$$

2- Let $S = \text{Soc}$ and consider Z_4 as Z -module. Since $S(Z_4) = \{\bar{0}, \bar{2}\}$, then $\frac{S_2(Z_4)}{S(Z_4)} =$

$$S\left(\frac{Z_4}{S(Z_4)}\right) = S\left(\frac{Z_4}{\{\bar{0}, \bar{2}\}}\right) \cong S(Z_2) = Z_2 = \frac{Z_4}{\{\bar{0}, \bar{2}\}} = \frac{Z_4}{S(Z_4)}. \text{ Thus } S_2(Z_4) = Z_4.$$

3- Let $S = Z$ and consider Z_8 as Z -module. Since $S(Z_8) = Z_8$, then $\frac{S_2(Z_8)}{S(Z_8)} =$

$$S\left(\frac{Z_8}{S(Z_8)}\right) = S\left(\frac{Z_8}{Z_8}\right) = S(0) = 0. \text{ Thus } S_2(Z_8) = S(Z_8) = Z_8.$$

4- Let M be a module. Then $S_2(M) = 0$ if and only if $S(M) = 0$.

Proof: Let $S_2(M) = 0$, for any module M . Since $S(M) \subseteq S_2(M)$, then $S(M) = 0$.

For the opposite, let $S(M) = 0$, for any module M . So $S_2(M) \cong \frac{S_2(M)}{0} = \frac{S_2(M)}{S(M)} =$

$$S\left(\frac{M}{S(M)}\right) = S\left(\frac{M}{0}\right) \cong S(M) = 0. \text{ Thus } S_2(M) = 0.$$

5- Let M be a module. If $S(M) = M$, then $S_2(M) = M$. The converse is true if S is a radical property.

Proof: Let $S(M) = M$, for any module M . Since $\frac{S_2(M)}{S(M)} = S\left(\frac{M}{S(M)}\right) = S\left(\frac{M}{M}\right) = S(0) = 0$, then $\frac{S_2(M)}{S(M)} = 0$. Thus $S_2(M) = S(M) = M$.

For the converse, let $S_2(M) = M$, for any module M . Since S is a radical property, then $\frac{S_2(M)}{S(M)} = S\left(\frac{M}{S(M)}\right) = 0$. Thus $S(M) = S_2(M) = M$.

Proposition 2.3

Let M_1 be a module. Then $S\left(\frac{M}{S_2(M)}\right) = 0$ if and only if S_2 is a radical property.

Proof: Let $S\left(\frac{M}{S_2(M)}\right) = 0$, for any module M . Then $S_2\left(\frac{M}{S_2(M)}\right) = 0$, by Rem. 2.2-4. Thus S_2 is a radical property.

For the opposite, let S_2 be a radical property. So $S_2\left(\frac{M}{S_2(M)}\right) = 0$, for any module M . Thus $S\left(\frac{M}{S_2(M)}\right) = 0$, by Rem. 2.2-4.

Proposition 2.4

Let M be a module. Then $S(S_2(M)) = S(M)$.

Proof: Clearly that $S(S_2(M)) \subseteq S(M)$. Since $S(M) \subseteq S_2(M)$, then $S(S(M)) \subseteq S(S_2(M))$. But $S(S(M)) = S(M)$, therefore $S(M) \subseteq S(S_2(M))$. Thus $S(S_2(M)) = S(M)$.

Proposition 2.5

Let M be a module. Then $S_2(S_2(M)) = S_2(M)$.

Proof: Since $\frac{S_2(S_2(M))}{S(S_2(M))} = S\left(\frac{S_2(M)}{S(S_2(M))}\right)$ and $S(S_2(M)) = S(M)$ by Prop. 2.3, then $\frac{S_2(S_2(M))}{S(S_2(M))} = S\left(\frac{S_2(M)}{S(M)}\right)$. But $S\left(\frac{S_2(M)}{S(M)}\right) = S\left(S\left(\frac{M}{S(M)}\right)\right) = S\left(\frac{M}{S(M)}\right) = \frac{S_2(M)}{S(M)}$, therefore $S_2(S_2(M)) = S_2(M)$.

Proposition 2.6

Let M and N be two modules and let $f: M \rightarrow N$ be a homomorphism, then $f(S_2(M)) \subseteq S_2(N)$.

Proof: Let $g: \frac{M}{S(M)} \rightarrow \frac{N}{S(N)}$ be a map defined by $g(x + S(M)) = f(x) + S(N)$, $\forall x \in M$.

To show that g is well define. Let $x_1 + S(M) = x_2 + S(M)$, $x_1, x_2 \in M$, then $x_1 - x_2 \in S(M)$ and hence $f(x_1) - f(x_2) = f(x_1 - x_2) \in S(N)$, by [2]. Therefore $f(x_1) + S(N) = f(x_2) + S(N)$. Clearly that g is a homomorphism. By [2]

$g\left(S\left(\frac{M}{S(M)}\right)\right) \subseteq S\left(\frac{N}{S(N)}\right)$. Since $S\left(\frac{M}{S(M)}\right) = \frac{S_2(M)}{S(M)}$ and $\left(\frac{N}{S(N)}\right) = \frac{S_2(N)}{S(N)}$, then $g\left(\frac{S_2(M)}{S(M)}\right) \subseteq \frac{S_2(N)}{S(N)}$.

Hence $\frac{f(S_2(M)) + S(N)}{S(N)} \subseteq \frac{S_2(N)}{S(N)}$. So $f(S_2(M)) + S(N) \subseteq S_2(N)$. But $S(N) \subseteq S_2(N)$, therefore $f(S_2(M)) \subseteq S_2(N)$.

Corollary 2.7

Let N be a submodule of a module M . Then $S_2(N) \subseteq S_2(M)$.

Proof: Let $i: N \rightarrow M$ be the inclusion map. Hence $i(S_2(N)) = S_2(N) \subseteq S_2(M)$, by Prop. 2.5.

Corollary 2.8

S_2 is a semiradical property.

Proof: Clear.

Remark 2.9

S_2 need not be a radical property. The following example illustrates:

The module Z_8 as Z -module. Let $S = Soc$, we know that $S(Z_8) = \{\bar{0}, \bar{4}\}$ and hence $\frac{S_2(Z_8)}{S(Z_8)} = S\left(\frac{Z_8}{S(Z_8)}\right) = S\left(\frac{Z_8}{\{\bar{0}, \bar{4}\}}\right) \cong S(Z_4) = Z_2 = \frac{\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}}{\{\bar{0}, \bar{4}\}} = \frac{\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}}{S(Z_8)}$. Thus $S_2(Z_8) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$.

Now $S_2\left(\frac{Z_8}{S_2(Z_8)}\right) = S_2\left(\frac{Z_8}{\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}}\right) \cong S_2(Z_2) = Z_2 \neq 0$.

Proposition 2.10

Let S be a radical property and M be a module. Then S_2 is a radical property.

Proof: Let M be a module. Hence $S\left(\frac{M}{S(M)}\right) = 0$, by our assumption. Therefore $\frac{S_2(M)}{S(M)} = S\left(\frac{M}{S(M)}\right) = 0$. So $S_2(M) = S(M)$. Thus S_2 is radical property.

Theorem 2.11

Let N be a module. Then $S(N) = 0$ if and only if $Hom(S_2(M), N) = 0$, for every module M .

Proof: $\Rightarrow$ Let $f: S_2(M) \rightarrow N$ be a homomorphism. By [2], $f(S(S_2(M))) \subseteq S(N) = 0$. Since $S(S_2(M)) = S(M)$, by Prop. 2.3, then $f(S(M)) = 0$. Hence $S(M) \subseteq \ker f$. By the first isomorphism theorem $\frac{S_2(M)}{\ker f} \cong f(S_2(M)) \subseteq N$. But $S(N) = 0$ by assumption, so $S\left(\frac{S_2(M)}{\ker f}\right) = 0$, by [2]. Now let $g: \frac{S_2(M)}{S(M)} \rightarrow \frac{S_2(M)}{\ker f}$ be a map defined by $g(x + S(M)) = x + \ker f, \forall x \in S_2(M)$. Clearly that g is an epimorphism. Since $S\left(\frac{S_2(M)}{S(M)}\right) = S\left(S\left(\frac{M}{S(M)}\right)\right) = S\left(\frac{M}{S(M)}\right) = \frac{S_2(M)}{S(M)}$, then $\frac{S_2(M)}{S(M)}$ has S . So $\frac{S_2(M)}{\ker f}$ has S and hence $S\left(\frac{S_2(M)}{\ker f}\right) = \frac{S_2(M)}{\ker f}$. But $S\left(\frac{S_2(M)}{\ker f}\right) = 0$, therefore $\frac{S_2(M)}{\ker f} = 0$ and hence $\ker f = S_2(M)$. Thus $Hom(S_2(M), N) = 0$.

$\Leftarrow$ Suppose that $Hom(S_2(M), N) = 0$, for every module M . Let $i: S_2(N) \rightarrow N$ be the inclusion map. Hence $i(S_2(N)) = S_2(N) = 0$. But $S(N) \subseteq S_2(N) = 0$, therefore $S(N) = 0$.

3-Extended S-Pure Submodule

Through this section introduces the concept of extended S -pure (es -pure) submodule. Basic properties of this type of modules are investigated. The text begins with the following definitions.

Definition 3.1

Let $N \leq M$. N is called extended S -pure submodule of M (briefly $N \leq_{esp} M$) if there is N_1 submodule of M such that $N \cap N_1 = 0$ and $S\left(\frac{M}{N \oplus N_1}\right) = 0$.

Definition 3.2

A short exact sequence $0 \rightarrow K \xrightarrow{f} N \xrightarrow{g} M \rightarrow 0$ is said to be es -pure sequence if $Im f \leq_{esp} N$. This case is referred to as f is es -pure monomorphism and g is es -pure epimorphism.

Remarks and examples 3.3

- 1- Let $S = S_{nr}$. Consider the module Z_4 as Z -module and let $N = \{\bar{0}, \bar{2}\}$. Since $\{0\}$ is the only submodule of Z_4 such that $N \cap 0 = 0$ and $S\left(\frac{Z_4}{N \oplus 0}\right) \cong S(Z_2) = 0$, then $N = \{\bar{0}, \bar{2}\} \leq_{esp} Z_4$.
- 2- Let $S = Z$. Consider the module $M = Z \oplus Z_4 \oplus Z_2$ as Z -module. Let $N = Z_4$, then there is $K = Z_2$ submodule of M such that $N \cap K = 0$ and $S\left(\frac{Z \oplus Z_4 \oplus Z_2}{Z_4 \oplus Z_2}\right) \cong S(Z) = 0$. Thus $N \leq_{esp} M$.
- 3- Let $S = S_a$. Consider the module Z_8 as Z -module and let $N = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$. Since $\{0\}$ is the only submodule of Z_8 such that $N \cap 0 = 0$ and $S\left(\frac{Z_8}{N \oplus 0}\right) \cong S(Z_2) = Z_2 \neq 0$, then $N = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ is not es -pure in Z_8 .
- 4- Let M be a module. Clearly that M and 0 are es -pure submodules of M .
- 5- Let $N \leq M$. If $N \leq_{s,p} M$, then $N \leq_{esp} M$.

Proof: Let $N \leq_{s,p} M$, then $S\left(\frac{M}{N}\right) = 0$. So $S\left(\frac{M}{N \oplus 0}\right) = S\left(\frac{M}{N}\right) = 0$. Thus $N \leq_{esp} M$.

The opposite is not generally true. For the example, let $S = Soc$ and consider Z_6 as Z -module. Then $N = \{\bar{0}, \bar{2}, \bar{4}\}$ is es -pure in Z_6 . Let $N_1 = \{\bar{0}, \bar{3}\}$ such that $N \cap N_1 = 0$ and $S\left(\frac{Z_6}{N \oplus N_1}\right) \cong S\left(\frac{Z_6}{Z_6}\right) = S(0) = 0$. But $S\left(\frac{Z_6}{N}\right) \cong S(Z_2) = Z_2 \neq 0$. Thus N is not s -pure in Z_6 .

- 6- Let M be an indecomposable module. Then $N \leq_{s,p} M$ if and only if $N \leq_{esp} M$.

Proof: $\Rightarrow$) Clear by (5).

$\Leftarrow$) Let $N \leq_{esp} M$. Then there is N_1 submodule of M such that $N \cap N_1 = 0$ and $S\left(\frac{M}{N \oplus N_1}\right) = 0$. But M is an indecomposable module, therefore $N_1 = 0$. So $S\left(\frac{M}{N \oplus N_1}\right) = S\left(\frac{M}{N}\right) = 0$. Thus $N \leq_{s,p} M$.

- 7- Let $N \leq M$. If $N \leq_{\oplus} M$, then $N \leq_{esp} M$.

Proof: Let $N = N \oplus N_1$. Then $S\left(\frac{M}{N \oplus N_1}\right) \cong S\left(\frac{M}{M}\right) = S(0) = 0$. Thus $N \leq_{esp} M$.

- 8- Let M be a semisimple module. Then each submodule of M is es -pure submodule of M .

Proof: Clear by (7).

Proposition 3.4

Let M and N be two modules. If $K \leq_{esp} M$ and $L \leq_{esp} N$, then $K \oplus L \leq_{esp} M \oplus N$

Proof: Suppose that $K \leq_{esp} M$ and $L \leq_{esp} N$. Then there is K_1 submodule of M such that $K \cap K_1 = 0$, $S\left(\frac{M}{K \oplus K_1}\right) = 0$ and L_1 submodule of N such that $L \cap L_1 = 0$,

$S\left(\frac{N}{L \oplus L_1}\right) = 0$. It's clear that $(K \oplus L) \cap (K_1 \oplus L_1) = 0$. Therefore $S\left(\frac{M \oplus N}{(K \oplus L) \oplus (K_1 \oplus L_1)}\right) = S\left(\frac{M \oplus N}{(K \oplus K_1) \oplus (L \oplus L_1)}\right) \cong S\left(\frac{M}{(K \oplus K_1)} \oplus \frac{N}{(L \oplus L_1)}\right) = S\left(\frac{M}{(K \oplus K_1)}\right) \oplus S\left(\frac{N}{(L \oplus L_1)}\right) = 0$, by [2]. Thus $K \oplus L \leq_{\text{esp}} M \oplus N$.

Proposition 3.5

Let N and K are submodules of M such that $N \leq K \leq M$. If $N \leq_{\text{esp}} M$ then $N \leq_{\text{esp}} K$.

Proof: Assume that $\leq_{\text{esp}} M$. Then there is $N_1 \leq M$ such that $N \cap N_1 = 0$ and $S\left(\frac{M}{N \oplus N_1}\right) = 0$. Let $N_2 = K \cap N_1$. It's clear that $N \cap (K \cap N_1) = 0$. So $\frac{K}{K \cap (N \oplus N_1)} \cong \frac{K + (N \oplus N_1)}{(N \oplus N_1)} \leq \frac{M}{(N \oplus N_1)}$, by the second isomorphism theorem. But $S\left(\frac{M}{N \oplus N_1}\right) = 0$, therefore $S\left(\frac{K}{K \cap (N \oplus N_1)}\right) = 0$. By the modular law, $S\left(\frac{K}{K \cap (N \oplus N_1)}\right) = S\left(\frac{K}{N \oplus (K \cap N_1)}\right) = 0$. Thus $N \leq_{\text{esp}} K$.

Proposition 3.6

Let N and K are submodules of M such that $N \leq K \leq M$. If $N \leq_{\text{esp}} K$ and $\leq_{\text{esp}} M$, then $N \leq_{\text{esp}} M$.

Proof: Suppose that $N \leq_{\text{esp}} K$ and $K \leq_{\text{esp}} M$. Then there is $N_1 \leq K$ such that $N \cap N_1 = 0$, $S\left(\frac{K}{N \oplus N_1}\right) = 0$ and $K_1 \leq M$ such that $K \cap K_1 = 0$, $S\left(\frac{M}{K \oplus K_1}\right) = 0$. Let $N_2 = N_1 \oplus K_1$. Claim that $N \cap (N_1 \oplus K_1) = 0$. To demonstrate that let $x \in N \cap (N_1 \oplus K_1)$. Hence $x = n$ and $x = n_1 \oplus k_1$. So $n = n_1 + k_1$ and hence $n - n_1 = k_1 \in K \cap K_1 = 0$. Therefore $n = n_1 \in N \cap N_1 = 0$ and hence $x = 0$. So $N \cap (N_1 \oplus K_1) = 0$. Since $\frac{K}{N \oplus N_1} \cong \frac{K}{N \oplus N_1} \oplus \frac{K_1}{K_1} \cong \frac{K \oplus K_1}{N \oplus N_1 \oplus K_1}$, then $S\left(\frac{K \oplus K_1}{N \oplus N_1 \oplus K_1}\right) = 0$. Consider the short exact sequence

$$0 \rightarrow \frac{K \oplus K_1}{N \oplus N_1 \oplus K_1} \xrightarrow{i} \frac{M}{N \oplus N_1 \oplus K_1} \xrightarrow{\pi} \frac{M}{K \oplus K_1} \rightarrow 0. \text{ But } S\left(\frac{K \oplus K_1}{N \oplus N_1 \oplus K_1}\right) = 0 \text{ and } S\left(\frac{M}{K \oplus K_1}\right) = 0, \text{ so } S\left(\frac{M}{N \oplus N_1 \oplus K_1}\right) = 0, \text{ by [2] Thus } N \leq_{\text{esp}} M.$$

Proposition 3.7

Let N and K be submodules of M such that $K \leq N \leq M$. If $N \leq_{\text{esp}} M$, then $\frac{N}{K} \leq_{\text{esp}} \frac{M}{K}$.

Proof: Let $N \leq_{\text{esp}} M$. Hence there is $N_1 \leq M$ such that $N \cap N_1 = 0$, $S\left(\frac{M}{N \oplus N_1}\right) = 0$.

Claim that $\frac{N \oplus N_1}{K} = \frac{N}{K} \oplus \frac{K \oplus N_1}{K}$. To illustrate that, let $x \in \frac{N \oplus N_1}{K}$, then $x = n + n_1 + K$ and hence $x = n + K + k + n_1 + K$. So $x \in \frac{N}{K} \oplus \frac{K \oplus N_1}{K}$ and hence $\frac{N \oplus N_1}{K} \subseteq \frac{N}{K} \oplus \frac{K \oplus N_1}{K}$.

Let $x \in \frac{N}{K} \oplus \frac{K \oplus N_1}{K}$, then $x = n + K + k + n_1 + K = n + n_1 + K$. So $x \in \frac{N \oplus N_1}{K}$ and hence $\frac{N}{K} \oplus \frac{K \oplus N_1}{K} \subseteq \frac{N \oplus N_1}{K}$. Therefore $\frac{N \oplus N_1}{K} = \frac{N}{K} \oplus \frac{K \oplus N_1}{K}$. To show that $\frac{N}{K} \cap \frac{K \oplus N_1}{K} = 0$. Let $x \in \frac{N}{K} \cap \frac{K \oplus N_1}{K}$, then $x = n + K$ and $x = k + n_1 + K = n_1 + K$. So $n + K = n_1 + K$ and hence $n - n_1 = k_1 \in K$. Hence $k_1 = k - n_1 \in K \cap N_1 = 0$ implies that $x = K$.

Therefore $\frac{N}{K} \cap \frac{K \oplus N_1}{K} = 0$. Now $S\left(\frac{\frac{M}{K}}{\frac{N}{K} \oplus \frac{K \oplus N_1}{K}}\right) = S\left(\frac{\frac{M}{K}}{\frac{N \oplus N_1}{K}}\right) = S\left(\frac{M}{N \oplus N_1}\right) = 0$. Thus

$$\frac{N}{K} \leq_{\text{esp}} \frac{M}{K}.$$

Theorem 3.8

Let $N \leq M$. If $N \leq_{\text{esp}} M$, then every S -Goldie module K is projective with regard to the short exact sequence $0 \rightarrow N \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{N} \rightarrow 0$. The converse is true if $S\left(\frac{\frac{M}{N}}{S_2(\frac{M}{N})}\right) = 0$, in particular S is a radical property.

Proof: Assume that $N \leq_{\text{esp}} M$. Then there is N_1 submodule of M such that $N \cap N_1 = 0$ and $S\left(\frac{M}{N \oplus N_1}\right) = 0$. Let K be S -Goldie module and let $f: K \rightarrow \frac{M}{N}$ be a homomorphism. Now consider the following diagram

$$\begin{array}{ccccccc}
 & & & K & & & \\
 & & & \downarrow f & & & \\
 & & h \swarrow & & \searrow & & \\
 0 & \rightarrow & N & \xrightarrow{i} & M & \xrightarrow{\pi} & \frac{M}{N} \rightarrow 0 \\
 & & & & \downarrow g & & \\
 & & & & \frac{M}{N \oplus N_1} & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}
 \quad \text{gof}$$

Where $g: \frac{M}{N} \rightarrow \frac{M}{N \oplus N_1}$ be a map defined by $g(m + N) = m + (N \oplus N_1), \forall x \in M$. Clearly that g is epimorphism. Since $S_2(K) = K$ and $S\left(\frac{M}{N \oplus N_1}\right) = 0$, then $\text{Hom}\left(K, \frac{M}{N \oplus N_1}\right) = 0$, by Theo. 2.11. Therefore $\text{gof}(K) = 0$. So $f(K) \subseteq \ker g$. But $\ker g = \{m + N : g(m + N) = m + (N \oplus N_1) = N \oplus N_1\}$
 $= \{m + N : m \in N \oplus N_1\}$
 $= \{n + n_1 + N : n \in N, n_1 \in N_1\} = \frac{N \oplus N_1}{N},$

by the second isomorphism theorem, $\frac{N \oplus N_1}{N} \cong \frac{N_1}{N \cap N_1} \cong N_1$. Since $\ker \pi = N$, then $\pi|_{N_1}: N_1 \rightarrow \frac{N \oplus N_1}{N}$ be an isomorphism. Hence we can define $(\pi|_{N_1})^{-1}: \frac{N \oplus N_1}{N} \rightarrow N_1$. Let $h = (\pi|_{N_1})^{-1} \circ f$. Thus $\pi \circ h = \pi \circ (\pi|_{N_1})^{-1} \circ f = f$.

For the converse, suppose that $S_2\left(\frac{M}{N}\right) = \frac{K}{N}$, K submodule of M . Clearly that $\frac{K}{N}$ is S -Goldie module, by prop. 2.4. Now consider the diagram

$$\begin{array}{ccccccc}
 & & & \frac{K}{N} & & & \\
 & & & \downarrow i_1 & & & \\
 & & h \swarrow & & \searrow & & \\
 0 & \rightarrow & N & \xrightarrow{i} & M & \xrightarrow{\pi} & \frac{M}{N} \rightarrow 0
 \end{array}$$

By our assumption there is $h: \frac{K}{N} \rightarrow M$ such that $\pi \circ h = i_1$. Therefore $\pi \circ h$ be a monomorphism. Let $N_1 = h\left(\frac{K}{N}\right)$, then $\pi(N_1) = \pi \circ h\left(\frac{K}{N}\right) = \frac{K}{N}$. Hence $\frac{N_1 + N}{N} = \frac{K}{N}$. Claim that $N \cap N_1 = 0$. Assume not and let $0 \neq x \in N \cap N_1$, then $x \in N, x \in N_1 = h\left(\frac{K}{N}\right)$. Let $x = h(k + N), k \in K$. Hence $k + N \neq N$. But $\pi(x) = \pi \circ h(k + N) = i_1(k + N) = k + N$, so $N = k + N$ which is a contradiction. So $\frac{N_1 \oplus N}{N} = \frac{K}{N}$. By the

third isomorphism theorem $\frac{M}{N \oplus N_1} \cong \frac{\frac{M}{N}}{\frac{N \oplus N_1}{N}} = \frac{\frac{M}{N}}{\frac{N}{K}}$. But $S_2\left(\frac{M}{N}\right) = \frac{K}{N}$, therefore $S\left(\frac{M}{N \oplus N_1}\right) \cong S\left(\frac{\frac{M}{N}}{S_2\left(\frac{M}{N}\right)}\right) = 0$. Thus $\leq_{esp} M$.

Proposition 3.9

Consider the following statements:

- 1- K *es*-pure in N .
- 2- Every S -Goldie module is projective with regard to the short exact sequence

$$0 \rightarrow K \xrightarrow{f} N \xrightarrow{g} M \rightarrow 0.$$

- 3- For any S -Goldie module K_1 , the sequence

$$0 \rightarrow \text{Hom}(K_1, K) \rightarrow \text{Hom}(K_1, N) \rightarrow \text{Hom}(K_1, M) \rightarrow 0 \text{ is exact.}$$

- 4- For any S -Goldie module K_1 , $\text{Hom}(K_1, g)$ is an epimorphism.

Then $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ and if S is a radical property, then $4 \Rightarrow 1$.

Proof: $1 \Rightarrow 2$) Let K_1 is S -Goldie module. Since K *es*-pure in N , Then K_1 is projective with regard to the short exact sequence $0 \rightarrow K \xrightarrow{f} N \xrightarrow{g} M \rightarrow 0$, by Theo. 3.7.

$2 \Rightarrow 3$) Clear.

$3 \Rightarrow 4$) Let K_1 is S -Goldie module. Consider the short exact sequence $0 \rightarrow K \xrightarrow{f} N \xrightarrow{g} M \rightarrow 0$. Thus $\text{Hom}(K_1, g)$ is an epimorphism, by our assumption.

$4 \Rightarrow 1$) Let $K \leq N$ and let K_1 be S -Goldie module. Consider the short exact sequence $0 \rightarrow K \xrightarrow{f} N \xrightarrow{g} M \rightarrow 0$. Hence $\text{Hom}(K_1, g)$ is an epimorphism, by our assumption. But S is a radical property, So K_1 is projective with regard to the short exact sequence. Thus $\leq_{esp} N$, by Theo. 3.7.

Proposition 3.10

The following statements are equivalent:

- 1- Every module M with $S(M) = 0$ is projective.
- 2- For every module M , every *es*-pure submodule of M is a direct summand.

Proof: $1 \Rightarrow 2$ Let $N \leq_{esp} M$. Hence there is $N_1 \leq M$ such that $N \cap N_1 = 0$ and $S\left(\frac{M}{N \oplus N_1}\right) = 0$. By our assumption $\frac{M}{N \oplus N_1}$ is projective. Hence the following short exact sequence is split

$$0 \rightarrow N \oplus N_1 \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{N \oplus N_1} \rightarrow 0. \text{ So } N \oplus N_1 \leq_{\oplus} M. \text{ Thus } \leq_{\oplus} M.$$

$2 \Rightarrow 1$ Let M be a module such that $S(M) = 0$. Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} M \rightarrow 0$ be a short exact sequence. By the first isomorphism Theorem, $\cong \frac{B}{\ker g}$. Now consider the short exact sequence

$0 \rightarrow f(A) \xrightarrow{i} B \xrightarrow{\pi} \frac{B}{f(A)} \rightarrow 0$. But $M \cong \frac{B}{\ker g} = \frac{B}{f(A)}$ and $S(M) = 0$, therefore $S\left(\frac{B}{f(A)}\right) = 0$. Hence $f(A)$ is *s*-pure in B . Therefore $f(A)$ is *es*-pure in B , by Rem. 3.2-5. By our

assumption $f(A) \leq_{\oplus} B$. So the short exact sequence is split. Thus M_1 is a projective.

Recall that A module M is said to be *s.p-Extended* (*s.p.CS-module* for short) if every *s-pure* submodule of M is a direct summand of M .

Proposition 3.11

Let M be a module. Then M is a *s.p.CS-module* if and only if every *es-pure* submodule of M is a direct summand of M .

Proof: $\Leftarrow$) It is clear.

$\Rightarrow$) Let M is *s.p.CS-module* and $N \leq_{esp} M$. Hence there is N_1 submodule of M such that $N \cap N_1 = 0$ and $S(M) = 0$. Hence $N \oplus N_1$ be *s-pure* submodule of M and so $N \oplus N_1 \leq_{\oplus} M$ by our assumption. Thus $\leq_{\oplus} M$.

4-es-flat Modules

This section aims to delve into the concept of *es-flat* module. Some basic properties of this class of modules are investigated. When $S = Z$, the concepts *es-flat* and *ec-flat* are coincide. The text begins with the definitions.

Definition 4.1

A module M is said to be *es-flat* module if for any epimorphism $f: N \rightarrow M$, $\ker f \leq_{esp} N$.

We infer that, when S is radical property M is *es-flat* module if and only if $Hom(K, N) \rightarrow Hom(K, M)$ is epimorphism, for each S -Goldie module K , by Theo. 3.7.

Remarks and Examples 4.2

1- Let M_1 be a module. If $S(M) = 0$, then M is *es-flat* module.

Proof: Let $f: N \rightarrow M$ be an epimorphism. By the first isomorphism Theorem, $\frac{N}{\ker f} \cong M$. Then $S\left(\frac{N}{\ker f}\right) = 0$. Hence $\ker f$ is *s-pure* in N . So $\ker f$ is *es-pure* in N , by Rem. 3.2-5. Thus M is *es-flat* module.

For example, let $S = Snr$. The module Z_4 as Z -module is *es-flat* module, where $S(Z_4) = 0$.

2- Every projective module is *es-flat* module.

Proof: Suppose that M be a projective module and let $f: N \rightarrow M$ be an epimorphism. Consider the .e.s. $0 \rightarrow \ker f \xrightarrow{i} N \xrightarrow{f} M \rightarrow 0$. Since M is a projective module, then the s.e.s. is split. Therefore $\ker f \leq_{\oplus} N$. So $\ker f$ is *es-pure* in N , by Rem. 3.2-6. Thus M is *es-flat* module.

3- Let $S = Soc$. The modules Z_6 as Z -module is not *es-flat*. To show that, let $f: Z \rightarrow Z_6$ be a map defined by $f(x) = \bar{x}, \forall x \in Z$. Clearly that f is an epimorphism and $\ker f = 6Z$. Since Z is an indecomposable, then $\{0\}$ is the only submodule of Z such that $6Z \cap \{0\} = 0$. Hence $S\left(\frac{Z}{6Z \oplus 0}\right) \cong S\left(\frac{Z}{6Z}\right) \cong S(Z_6) = Z_6 \neq 0$. So $\ker f = 6Z$ is not *es-pure* submodule of Z . Thus Z_6 is not *es-flat* module.

4- *es-flat* module need not be a flat module. For example, let $S = Snr$ and consider the module Z_2 as Z -module. Since $S(Z_2) = 0$, then Z_2 is *es-flat* module, by (1). But Z is an integral domain and Z_2 is a torsion module, therefore Z_2 is not a flat module, by [9], Prop. 3.49, p.134].

5- A flat module need not be *es*-flat. The example illustrate, Z_4 as Z_4 -module is free and hence Z_4 is flat, by [5], Theo. 10.4.4, p.259]. Let $S = Soc$ and let $f: Z \rightarrow Z_4$ be a map defined by $f(x) = \bar{x}, \forall x \in Z$. Clearly that f is an epimorphism and $\ker f = 4Z$. One can easily show that $4Z$ is not *es*-pure in Z . Thus Z_4 is not *es*-flat module.

6- If M is *es*-flat module, then M need not be *ec*-flat module. For example, let $S = Snr$. Clearly that the module Z_2 as Z -module is *es*-flat module, by (1).

Now let $\pi: Z \rightarrow Z_2$ be the natural epimorphism. Clearly that $\ker \pi = 2Z$ and $K = \{0\}$ is the only submodule of Z such that $2Z \cap \{0\} = 0$. But $\frac{Z}{2Z \oplus 0} = \frac{Z}{2Z} \cong Z_2$ is not nonsingular, therefore $2Z$ is not *ec*-pure submodule of Z . Thus Z_2 is not *ec*-flat module.

Before we state our next result, we need to recall the following definition, let $f_1: M_1 \rightarrow M$ and $f_2: M_2 \rightarrow M$ be two morphisms. A triple (N, h_1, h_2) is called a pullback for the pair (f_1, f_2) . If the diagram

$$\begin{array}{ccc} N & \xrightarrow{h_2} & M_2 \\ h_1 \downarrow & & \downarrow f_2 \\ M_1 & \xrightarrow{f_1} & M \end{array}$$

is commute and for every pair of morphisms $g_1: K \rightarrow M_1$ and $g_2: K \rightarrow M_2$ such that $f_1 \circ g_1 = f_2 \circ g_2$, there is a existence unique morphism $g: K \rightarrow N$ such that $h_1 \circ g = g_1$ and $h_2 \circ g = g_2$, see [11], 10.1, p.73].

Theorem 4.3

Let S be a radical property and let M be a module. Then the following statements are equivalent:

- 1- M is *es*-flat module.
- 2- There is an *es*-pure sequence $0 \rightarrow K \rightarrow N \rightarrow M \rightarrow 0$ with N is a projective.
- 3- There is an *es*-pure sequence $0 \rightarrow K \rightarrow N \rightarrow M \rightarrow 0$ with N *es*-flat.

Proof: $1 \Rightarrow 2$) Assume that M is *es*-flat module. Hence there is a existence free module N and an epimorphism $f: N \rightarrow M$, by [5] Coro. 4.4.4, p.89]. Hence N is a projective, by [5], Theo. 4.4.6, p.90]. Since M be *es*-flat module, then $\ker f$ is *es*-pure submodule of N .

$2 \Rightarrow 3$) Clear, by Rem. 4.2-2.

$3 \Rightarrow 1$) Let $f_1: N_1 \rightarrow M$ be an epimorphism. By our assumption, there is a existence an *es*-pure sequence $0 \rightarrow K \rightarrow N \xrightarrow{f} M \rightarrow 0$ with N *es*-flat module. Let (N, g, g_1) be the pullback of f and f_1 . We have the following commute diagram:

$$\begin{array}{ccccc} N & \xrightarrow{g_1} & N & \rightarrow & 0 \\ g \downarrow & & \downarrow f & & \\ N_1 & \xrightarrow{f_1} & M & \rightarrow & 0 \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

Since N *es*-flat module, then g_1 is an *es*-pure epimorphism. But f is an *es*-pure epimorphism, therefore $f_1 \circ g = f \circ g_1$ is also an *es*-pure epimorphism, by [11], 33.2,

p.275]. Hence f_1 is an *es*-pure epimorphism, by [11], 33.2, p.275]. Therefore $\ker f_1$ is *es*-pure submodule of N_1 . Thus M is *es*-flat module.

Proposition 4.4

Let S be a radical property and let $0 \rightarrow K \xrightarrow{f} N \xrightarrow{g} M \rightarrow 0$ be the short exact sequence. If the short exact sequence is *es*-pure sequence and N is *es*-flat module, then M is *es*-flat module.

Proof: Suppose that N be *es*-flat module. Hence there exist *es*-pure sequence

$0 \rightarrow K_1 \xrightarrow{f_1} F \xrightarrow{g_1} N \rightarrow 0$ with F *es*-flat, by Theo. 4.3. Now consider the short exact sequence

$0 \rightarrow \ker g \circ g_1 \xrightarrow{i} F \xrightarrow{g \circ g_1} M \rightarrow 0$. But g and g_1 are *es*-pure epimorphisms, so $g \circ g_1$ is *es*-pure epimorphism, by [11], 33.2, p.275]. Therefore $g \circ g_1$ is *es*-pure sequence with F *es*-flat. Thus M is *es*-flat module, by Theo. 4.3.

Proposition 4.5

Let S be a radical property and let $0 \rightarrow K \xrightarrow{f} N \xrightarrow{g} M \rightarrow 0$ be the short exact sequence. If K and M are *es*-flat modules, then N is *es*-flat module.

Proof: Suppose that K and M are *es*-flat modules and let $g_1: N_1 \rightarrow N$ be an epimorphism. We have the following commute diagram:

$$\begin{array}{ccccccc}
 0 & \rightarrow & \ker g \circ g_1 & \xrightarrow{i} & N_1 & \xrightarrow{g \circ g_1} & M \rightarrow 0 \\
 & & \downarrow f_1 & & \downarrow g_1 & & \downarrow I \\
 0 & \longrightarrow & K & \xrightarrow{f} & N & \xrightarrow{g} & M \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

For each S -Goldie module K_1 , we apply the functor $\text{Hom}(K_1, -)$ and obtain the diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & \text{Hom}(K_1, \ker g \circ g_1) & \rightarrow & \text{Hom}(K_1, N_1) & \rightarrow & \text{Hom}(K_1, M) \\
 & & \downarrow \text{Hom}(K_1, f_1) & & \downarrow \text{Hom}(K_1, g_1) & & \downarrow \text{Hom}(K_1, I) \\
 0 & \longrightarrow & \text{Hom}(K_1, K) & \longrightarrow & \text{Hom}(K_1, N) & \longrightarrow & \text{Hom}(K_1, M) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Since K and M are *es*-flat modules, then $\text{Hom}(K_1, f_1)$ and $\text{Hom}(K_1, I)$ are epimorphisms, for each S -Goldie module K_1 , by Theo. 3.7. Therefore $\text{Hom}(K_1, g_1)$ is epimorphism, by [11], 36.1, p.304]. Thus N is *es*-flat module, by Theo. 3.7.

Proposition 4.6

Let S be a radical property. Then a direct summand of *es*-flat is *es*-flat module.

Proof: Assume that M be *es*-flat module and let $N \leq_{\oplus} M$. Then there exist *es*-pure sequence $0 \rightarrow K \xrightarrow{f} F \xrightarrow{g} M \rightarrow 0$ with F *es*-flat, by Theo. 4.3. Let $P: M \rightarrow N$ be the projection map. Since $\ker P \leq_{\oplus} M$, then $\ker P$ is *es*-pure in M by Rem. 3.2-7. Hence $M \xrightarrow{P} N \rightarrow 0$ is *es*-pure epimorphism. Consider the short exact sequence $0 \rightarrow \ker P \circ g \rightarrow F \xrightarrow{P \circ g} N \rightarrow 0$. But g and P are *es*-pure epimorphisms, so $P \circ g$ is *es*-

pure epimorphism, by [11], 33.2, p.275]. Therefore Pog is es -pure sequence with F es -flat. Thus N is es -flat module, by Theo. 4.3.

Proposition 4.7

Let S be a radical property and let $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 . If M_1 and M_2 are es -flat modules, then M is es -flat module.

Proof: Suppose that M_1 and M_2 are es -flat modules. Then there exists an es -pure sequences $0 \rightarrow K_1 \rightarrow N_1 \rightarrow M_1 \rightarrow 0$ and $0 \rightarrow K_2 \rightarrow N_2 \rightarrow M_2 \rightarrow 0$ with N_1 and N_2 are projective, by Theorem 4.3. Consider the short exact sequence $0 \rightarrow K_1 \oplus K_2 \rightarrow N_1 \oplus N_2 \rightarrow M_1 \oplus M_2 \rightarrow 0$. Since $K_1 \leq_{esp} N_1$ and $K_2 \leq_{esp} N_2$, then $K_1 \oplus K_2 \leq_{esp} N_1 \oplus N_2$, by Prop. 3.3. But N_1 and N_2 are projective, therefore $N_1 \oplus N_2$ is a projective. Thus $M = M_1 \oplus M_2$ is es -flat module, by Theo. 4.3.

Corollary 4.8

Let S be a radical property and let $M = \bigoplus_{i \in I} M_i$ be a direct sum of submodules M_i , $\forall i \in I$. Then M is es -flat if and only if M_i is es -flat, $\forall i \in I$.

Proof: Clear by prop. 4.6. and 4.7. by induction.

Proposition 4.9

Let M_1 be a module. A S -Goldie K is es -flat if and only if K is a projective.

Proof: Let K is S -Goldie and es -flat module. Hence K is projective with regard to es -pure sequence, by Theo. 3.7.

$$0 \rightarrow \ker f \xrightarrow{i} M_1 \xrightarrow{f} K \rightarrow 0$$

$\begin{array}{c} \nearrow g \\ \searrow f \end{array} \downarrow$

Therefore there is $g: K \rightarrow M_1$ such that $fog = I$. Therefore the short exact sequence has a right inverse and hence split. Thus K is a projective.

The converse is clear by Rem. 4.2-2.

Proposition 4.10

Let $S(R) = 0$. Then M is es -flat if and only if $S(M) = 0$.

Proof: Assume that M is es -flat module. Hence by Theo. 3.7, $S_2(M)$ is projective with regard to the short exact sequence

$$0 \rightarrow K \xrightarrow{i} F \xrightarrow{f} M \rightarrow 0$$

$\begin{array}{c} S_2(M) \\ \nearrow g \\ \searrow f \end{array} \downarrow j$

so there is a existence morphism $g: S_2(M) \rightarrow F$ such that $fog = j$. Hence g is a monomorphism and hence $S_2(M) \cong F \cong \bigoplus_i R$, by [5], Lem.4.4.1, p.88]. So $(M) = S(S_2(M)) \cong S(\bigoplus_i R) = \bigoplus_i S(R) = 0$, by Prop. 2.3. Thus $S(M) = 0$.

The converse is clear by Remark 4.2-1.

Proposition 4.11

Let $S(R) = R$. Then M is es -flat if and only if M is a projective.

Proof: Let M be es -flat. Hence there is a existence an es -pure sequence $0 \rightarrow K \rightarrow N \rightarrow M \rightarrow 0$ with N is a projective, by Theo. 4.3. So $K \leq_{esp} N$ and hence there exists

$K_1 \leq N$ such that $K \cap K_1 = 0$ and $S\left(\frac{N}{K \oplus K_1}\right) = 0$. Since $N \cong \oplus_i R$, by [5], Lem.4.4.1, p.88] then $S(N) \cong S(\oplus_i R) = \oplus_i S(R) = \oplus_i R \cong N$ and hence N has S . Now let $\pi: N \rightarrow \frac{N}{K \oplus K_1}$. Hence $\frac{N}{K \oplus K_1}$ has S such that $S\left(\frac{N}{K \oplus K_1}\right) = \frac{N}{K \oplus K_1}$. But $S\left(\frac{N}{K \oplus K_1}\right) = 0$, therefore $N = K \oplus K_1$ and hence $K \leq_{\oplus} N$. So the short exact sequence is split. Thus M is a projective.

The converse is clear by Rem. 2.4-2.

Proposition 4.12

Let $f: N \rightarrow M$ be an epimorphism. If M is es -flat module, then any S -Goldie submodule of M is an isomorphic to S -Goldie submodule of N . In particular, $S_2(M)$ embeds in a projective module.

Proof: Let K be S -Goldie submodule of M and let $i: K \rightarrow M$. Since M is es -flat module, then that $Hom(K, N) \rightarrow Hom(K, M) \rightarrow 0$ is exact. Hence there is a existence morphism $g: K \rightarrow N$ such that $f \circ g = i$. So g is a monomorphism and S -Goldie modules are closed under homomorphic image. Therefore $g(K)$ is S -Goldie submodule of N . The specific case follows by taking an epimorphism $f_1: F \rightarrow K$ with F a projective and $K = S_2(M)$.

Proposition 4.13

Every cyclic module M with $S(M) = 0$ is a projective if and only if every cyclic es -flat module is a projective.

Proof: Assume that M is cyclic es -flat module. Then $\cong \frac{R}{I}$, for some ideal I of R . Since $\frac{R}{I}$ is es -flat module, then I is es -pure in R . So there is $J \leq R$ such that $I \cap J = 0$ and $S\left(\frac{R}{I \oplus J}\right) = 0$. Therefore $\frac{R}{I \oplus J}$ is projective by assumption. Now consider the short exact sequence $0 \rightarrow I \oplus J \xrightarrow{i} R \xrightarrow{\pi} \frac{R}{I \oplus J} \rightarrow 0$. Therefore the short exact sequence is split. So $I \oplus J \leq_{\oplus} R$ and hence $\leq_{\oplus} R$. Therefore the short exact sequence $0 \rightarrow I \rightarrow R \rightarrow \frac{R}{I} \rightarrow 0$ is split. Thus M is a projective.

For the converse, every module M with $S(M) = 0$ is es -flat module, by Rem. 4.2-1. Thus M is projective, by our assumption.

Proposition 4.14

Let S be a radical property. If every submodule of es -flat is es -flat, then $S(R)$ is a projective.

Proof: Since R as R -module is a projective, then R is es -flat, by Rem. 4.2-2. Hence $S(R)$ is es -flat by our assumption. But S radical property, therefore $S_2(R) = S(R)$. So $S_2(R)$ is projective with regard to es -pure sequence by Theo. 3.7. Thus $S(R)$ is projective by Prop. 4.7.

Proposition 4.15

Let S be a radical property and R be a ring. If every cyclic es -flat module is projective, then $R \cong M \oplus N$, where M is S -Goldie module and N is a projective.

Proof: Since R as R -module is a projective, then R is es -flat, by Rem. 4.2-2. But S is radical property, therefore $S\left(\frac{R}{S_2(R)}\right) = 0$. Hence $\frac{R}{S_2(R)}$ is es -flat, by Rem. 4.2-1.

Therefore $\frac{R}{S_2(R)}$ is a projective by our assumption. So the short exact sequence $0 \rightarrow S_2(R) \rightarrow R \rightarrow \frac{R}{S_2(R)} \rightarrow 0$ is split.

$$\text{Thus } \cong S_2(R) \oplus \frac{R}{S_2(R)}, \text{ where } M = S_2(R) \text{ and } N = \frac{R}{S_2(R)}.$$

Proposition 4.16

Let S be a radical property and M_1 be injective *s.p.* *CS*-module. Then M is *es*-flat if and only if $S_2(M)$ is a projective.

Proof: Let M is *es*-flat. Then $S_2(M)$ embeds in a projective module say F , by Prop. 4.10. But S be a radical property, therefore $S\left(\frac{M}{S_2(M)}\right) = 0$ and hence $S_2(M)$ is *s*-pure in M . So $S_2(M) \leq_{\oplus} M$ by our assumption. Since M is an injective, then $S_2(M)$ is an injective. So the short exact sequence $0 \rightarrow S_2(M) \rightarrow F \xrightarrow{F} \frac{F}{S_2(M)} \rightarrow 0$ is split. Thus $S_2(M)$ is a projective.

For the converse, assume that $S_2(M)$ is a projective. Let $0 \rightarrow K \rightarrow F \xrightarrow{g} M \rightarrow 0$ be a short exact sequence with F a projective. Now let L be *S*-Goldie module and let $f: L \rightarrow M$, then $f(L)$ embeds in $S_2(M)$ that is there is an inclusion $i: f(L) \rightarrow S_2(M)$. There is also an inclusion $i_1: f(L) \rightarrow M$. Now we obtain the diagram:

$$\begin{array}{ccc} & L & \\ & \downarrow f_1 & \\ f(L) & \xrightarrow{i} & S_2(M) \\ & \downarrow i_1 & \\ 0 \rightarrow K \rightarrow F & \xrightarrow{g} & M \rightarrow 0 \end{array}$$

By the injectivity of M , there is a homomorphism $g_1: S_2(M) \rightarrow M$ such that $g_1 \circ i = i_1$. Hence by the projectivity of $S_2(M)$, there exists a homomorphism $g_2: S_2(M) \rightarrow F$ such that $g \circ g_2 = g_1$. Taking $h = g_2 \circ i \circ f_1: L \rightarrow F$. We get $g \circ h = f$. Thus M is *es*-flat module, by Theo. 4.3.

Proposition 4.17

Let S be a radical property and M be a module. Then the following statements are equivalent:

- 1- Every module M with $S(M) = 0$ is projective.
- 2- Every *es*-flat module is projective.
- 3- Every *s*-pure submodule of a projective module is a direct summand.

Proof: $1 \Rightarrow 2$) Let M is *es*-flat. Hence there is existence an *es*-pure sequence $0 \rightarrow K \rightarrow N \rightarrow M \rightarrow 0$, with N a projective, by Theo. 4.3. So, there is $K_1 \leq N$ such that $K \cap K_1 = 0$ and $S\left(\frac{N}{K \oplus K_1}\right) = 0$. Hence $\frac{N}{K \oplus K_1}$ is a projective by our assumption.

Consider the short exact sequence $0 \rightarrow K \oplus K_1 \xrightarrow{i} N \xrightarrow{\pi} \frac{N}{K \oplus K_1} \rightarrow 0$. So the sequence is split and hence $K \oplus K_1 \leq_{\oplus} N$. Therefore $\leq_{\oplus} N$. So the short exact sequence $0 \rightarrow K \rightarrow N \rightarrow M \rightarrow 0$ is split. Thus M is a projective.

$2 \Rightarrow 3$) Let N be a projective module and let $K \leq_{s,p} N$. Then $K \leq_{esp} N$, by Rem. 3.2-4. Since we have es -pure sequence $0 \rightarrow K \xrightarrow{i} N \xrightarrow{\pi} \frac{N}{K} \rightarrow 0$ with N a projective. Hence $\frac{N}{K}$ is es -flat module, by Theo. 4.3. Therefore $\frac{N}{K}$ is a projective by our assumption. So the sequence is split. Thus $\leq_{\oplus} N$.

$3 \Rightarrow 1$) Suppose that M is a module with $S(M) = 0$. Hence there is a free module N and an eipmorphism $f: N \rightarrow M$ by [5] Coro. 4.4.4, p.89]. Consider the short exact sequence $0 \rightarrow K \rightarrow N \rightarrow M \rightarrow 0$. Since $\frac{N}{K} \cong M$, by the first isomorphism theorem, then $S\left(\frac{N}{K}\right) \cong S(M) = 0$. So K is s -pure in N . But N is free and hence a projective, therefore $K \leq_{\oplus} N$ by our assumption. Thus M is a projective module.

References

- [1] F. W. Anderson, K. R. Fuller, *Rings, Modules and Homomorphisms*, in: F.W. Anderson, K.R. Fuller (Eds.) *Rings and Categories of Modules*, Springer New York, New York, NY, 1992, 10-64.
- [2] Y. Durgun, S. Ozdemir, *On S -closed submodules*, J. Korean Math. Soc **54** (2017), 1281-1299.
- [3] Y. Durgun, S. Özdemir, *On τ -closed submodules*, Proceedings - Mathematical Sciences **130** (2020), 1.
- [4] K. R. Goodearl, *Singular torsion and the splitting properties*, American Mathematical Society, 1972.
- [5] F. Kasch, *Modules and rings*, Academic press, 1982.
- [6] J. T. Knight, *Commutative algebra*, Cambridge University Press, 1971.
- [7] D. Yılmaz, Ç. Ayşe, *Proper classes generated by τ -closed submodules*, Analele Stiintifice ale Universitatii "Ovidius" Constanta. Seria Matematica **27** (2019), 83-95.
- [8] N. Hamad, B. Al-Hashimi, *Some Results on the Jacobson Radicals and the M -Radicals*, Basic Sciences and Engineering **1** (2002).
- [9] J. J. Rotman, *An introduction to homological algebra*, Springer, 2009.
- [10] K. Goodearl, *Ring theory: Nonsingular rings and modules*, CRC Press, 1976.
- [11] R. Wisbauer, *Foundations of module and ring theory*, Routledge, 2018.