

## TRISECTION OF THE ANGLE USING ONLY LINES AND CIRCLES

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**Abstract:** *The current paper shows step-by-step how to trisect an angle. The trichotomized angle is formed from the opposite straight segment to the existing one, which is equal to  $\pi^{1/\pi}$ .*

### Introduction

The history of the problem of trisection of the angle is rooted in the ancient Hellenic world, as a mathematical, physical, and philosophical problem. This paper shows proof of it. The structure of this research is divided into three parts: introduction, proof, and conclusion. Trisection of an angle is a fundamental concept in classical geometry, which has a long history in mathematics dating back to the days of ancient Greek mathematicians like Hippias of Elis and Archimedes [1, 3-6, 8]. The problem is to divide any given angle into three equal parts using only two instruments: a straightedge and a compass. Though angles can be trisected using classical methods, the problem has remained unsolved for more than two millennia [2, 7]. This problem is considered a turning point in the history of mathematics, changing the focus from geometric construction to arithmetic structure. It is one of the three universal problems, even before space-time, like the Delian problem and squaring the circle. In this context, the interpretation of classical geometric problems can be extended beyond their historical formulation to a deeper ontological level [9-12]. Mathematical structures may be understood as elements of a broader axiomatics of physics, where fundamental principles exist independently of empirical space-time constraints. From this perspective, problems such as the trisection of the angle, the duplication of the cube, and the squaring of the circle are not merely historical challenges but represent achronous constructs that precede physical reality. They reflect abstract entities that exist beyond space and time, indicating that mathematics serves as a formal language for expressing universal ideas that are inherently timeless. Thus, these classical problems may be understood as manifestations of pre-spatiotemporal logical structures existing in the space-time of ideas within the mathematical axiomatic physics [3].

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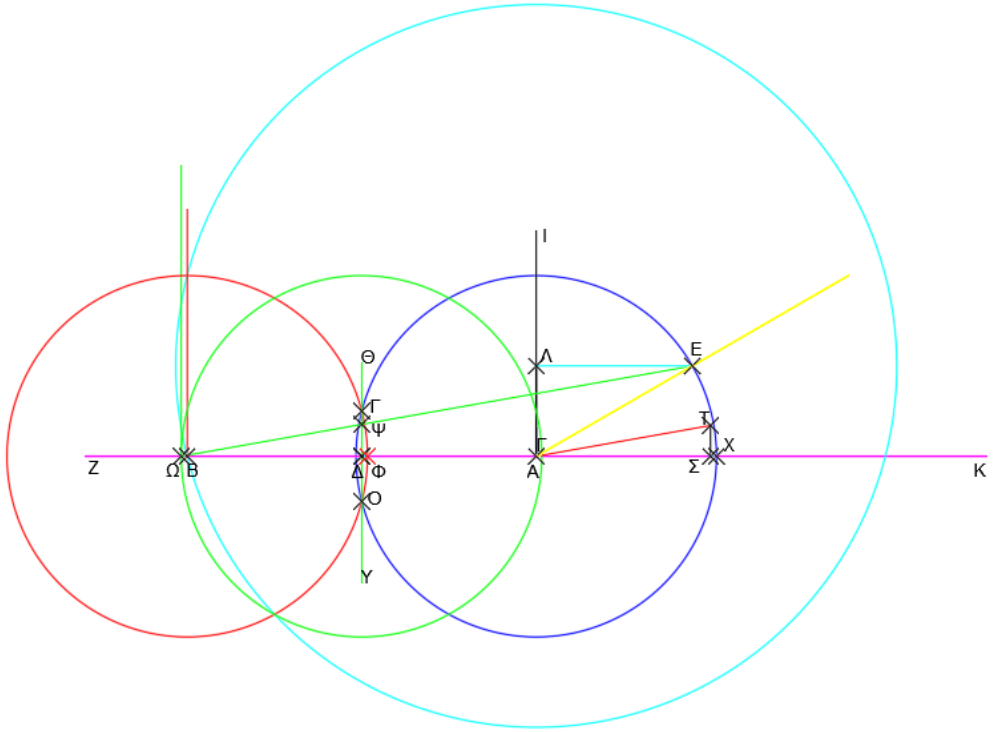
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**Proof****Figure 1:** Trisection of the angle (Author's scheme)

To trisect an angle

1. From point  $A$ , construct a line segment  $ZK$  (purple line).
2. Define an angle  $\angle KAE$  to be trisected.
3. With center  $A$  and radius  $AE$ , draw a circle (blue circle).
4. Construct a circle of radius  $2AE$  (cyan circle). The length  $2AE$  is obtained geometrically by placing two segments of length  $AE$  consecutively on a straight line. The center  $\Lambda$  is determined as the intersection of the line through  $E$  with the line  $AI$ .
5. Define point  $B$  as the intersection of the line  $ZK$  with the circle of radius  $2AE$ .
6. With center  $B$  and radius  $AE$ , draw a circle (red circle).



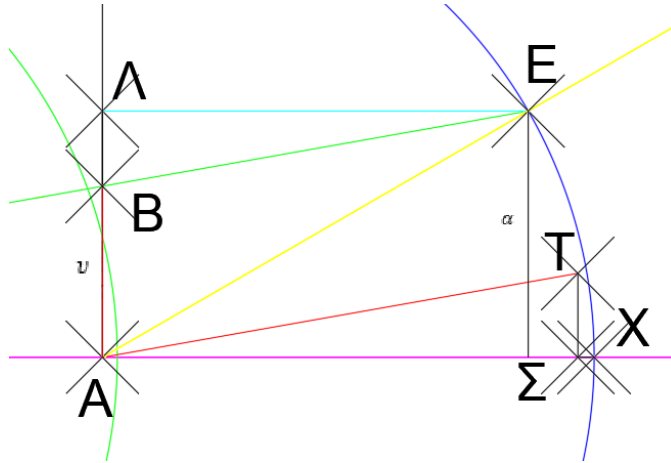


**Figure 3:** Trisection theorem (Author's scheme)

The trisection of an angle is achieved using only straight lines and circles, without any measurement.

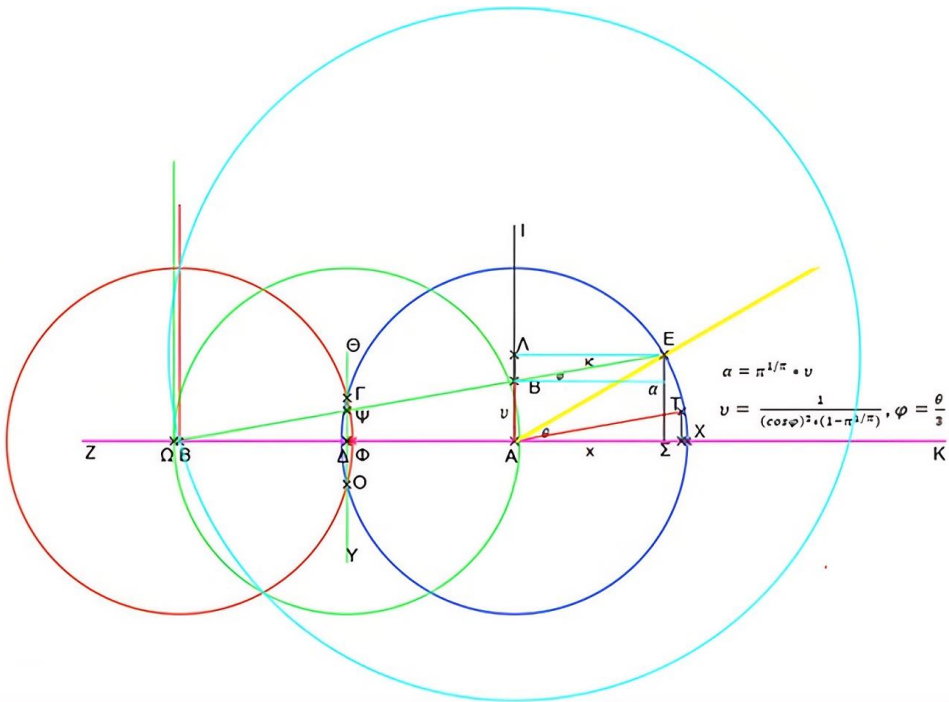


**Figure 4:** Trisection construction (Author's scheme)

For the trisection of the angle

$$\pi^{1/\pi} = \frac{\alpha}{v} \quad (1)$$

$$\alpha = \pi^{1/\pi} v \quad (2)$$

$$a = E\Sigma = \Lambda A, v = AB \quad (3)$$

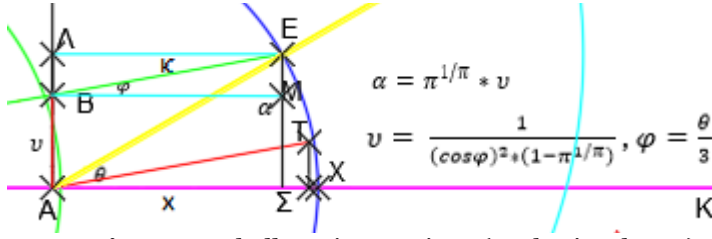


**Figure 5:** Challoumis equations (Author's scheme)

$$\kappa^2 = x^2(v - \alpha) \quad (4)$$

$$\kappa^2 = x^2(v - \pi^{1/\pi} v) \quad (5)$$

$$\kappa^2 = x^2 v (1 - \pi^{1/\pi}) \quad (6)$$

$$\cos \varphi = \frac{x}{\kappa} \Rightarrow x = \kappa \cos \varphi \quad (7)$$

$$\kappa^2 = \kappa^2 (\cos \varphi)^2 v (1 - \pi^{1/\pi}) \quad (8)$$

$$1 = (\cos \varphi)^2 v (1 - \pi^{1/\pi}) \quad (9)$$

$$v = \frac{1}{(\cos \varphi)^2 (1 - \pi^{1/\pi})}, \varphi = \frac{\theta}{3} \quad (10)$$

### Conclusion

The trichotomized angle is formed from the opposite straight segment to the existing one, which is equal to  $\pi^{1/\pi}$ .

### Analytical Expression of the Trisection Result

This section presents the analytical expression corresponding to the geometric construction.

From the configuration, the following relations are established

$$\kappa^2 = x^2(v - \alpha),$$

$$\cos \varphi = \frac{x}{\kappa},$$

and

$$\alpha = \pi^{1/\pi} v.$$

Substituting the proportional relation into the quadratic expression yields

$$\kappa^2 = x^2 v (1 - \pi^{1/\pi}).$$

Thus, it is obtained that

$$\cos^2 \varphi = \frac{x^2}{\kappa^2} = \frac{1}{v (1 - \pi^{1/\pi})}.$$

Hence, the length  $v$  is expressed as

$$v = \frac{1}{\cos^2 \varphi (1 - \pi^{1/\pi})}.$$

Since the geometric construction uniquely determines the configuration from the initial angle  $\theta$ , the resulting angle  $\varphi$  is fully characterized by these relations. Therefore, the analytical structure of the construction corresponds to an angle satisfying

$$\varphi = \frac{\theta}{3}.$$

### Conclusion

This paper presented a geometric construction for the trisection of an angle using only straight lines and circles. The construction is based on explicit geometric steps and relations derived from intersections of circles with equal radii and corresponding line segments.

In the geometric configuration, the angles

$$\angle E\Omega K, \angle \Lambda E\Omega, \angle KAT$$

are equal.

From the construction, the relevant points are defined as intersections of circles with equal radii and lines constructed through these points.

In particular, the segments involved are equal due to the radii of the circles used in the construction. Therefore, the corresponding triangles formed in the configuration are isosceles.

As a consequence, the base angles of these triangles are equal. By applying this property to the triangles determined by the points  $E, \Omega, K, \Lambda, E, \Omega$ , and  $K, A, T$ , it is obtained that

$$\angle E\Omega K = \angle \Lambda E\Omega = \angle KAT.$$

### Angle Trisection

Let  $\theta$  be the original angle. The construction produces an angle  $\varphi$  such that

$$\varphi = \frac{\theta}{3}.$$

From the previous lemma,

$$\angle E\Omega K = \angle \Lambda E\Omega = \angle KAT = \varphi.$$

Since these three equal adjacent angles compose the original angle  $\theta$ , it is obtained that

$$\theta = \varphi + \varphi + \varphi = 3\varphi.$$

Therefore,

$$\varphi = \frac{\theta}{3}.$$

The result demonstrates that the proposed geometric configuration yields a trisection of the given angle through a sequence of constructions involving only lines and circles. The equality of the constructed angles follows directly from the geometric properties of the configuration, leading to a consistent and well-defined subdivision of the original angle.

The angle equalities arise from the geometric construction, where the segments involved are radii of circles of equal length. Consequently, the triangles formed in

the configuration are isosceles, and their corresponding base angles are equal. This yields

$$\angle E\Omega K = \angle \Lambda E\Omega = \angle KAT.$$

### Theorem

Angle trisection is achieved when a geometric construction produces three equal adjacent angles whose sum equals the original angle  $\theta$ , implying  $\varphi = \theta/3$ .

## Appendix A. Formal Structure of the Trisection Construction

This appendix provides the complete formalization of the variables, relations, and lemmas underlying the proposed geometric construction for the trisection of an angle.

### Definitions and variables

Let the initial angle be defined as

$$\theta = \angle KAE.$$

It is aimed to construct an angle

$$\varphi = \frac{\theta}{3}.$$

Define the following geometric quantities

$$\alpha = E\Sigma = \Lambda A,$$

$$v = AB,$$

$$x = A\Sigma,$$

$$\kappa = BE.$$

Assume that the construction satisfies the proportional relation

$$\pi^{1/\pi} = \frac{\alpha}{v}.$$

Thus,

$$\alpha = \pi^{1/\pi} v.$$

The fundamental geometric relation derived from the construction is

$$\kappa^2 = x^2(v - \alpha).$$

Substituting for  $\alpha$

$$\kappa^2 = x^2 v (1 - \pi^{1/\pi}).$$

The trigonometric relation used in the construction is

$$\cos \varphi = \frac{x}{\kappa}.$$

### Lemma on proportional structure

Lemma A1. If

$$\pi^{1/\pi} = \frac{\alpha}{v},$$

then

$$\alpha = \pi^{1/\pi} v.$$

*Proof.* Direct multiplication yields the result.  $\square$

### Lemma on quadratic relation

Lemma A2. Under the construction relations,

$$\kappa^2 = x^2 v (1 - \pi^{1/\pi}).$$

*Proof.* Substitution of  $\alpha = \pi^{1/\pi} v$  into

$$\kappa^2 = x^2 (v - \alpha).$$

gives the result.  $\square$

### Lemma on derived length

Lemma A3. If

$$\cos \varphi = \frac{x}{\kappa},$$

then

$$v = \frac{1}{\cos^2 \varphi (1 - \pi^{1/\pi})}.$$

*Proof.* From

$$\cos^2 \varphi = \frac{x^2}{\kappa^2}.$$

It is obtained that

$$\frac{1}{\cos^2 \varphi} = \frac{\kappa^2}{x^2}.$$

Using Lemma A2,

$$\frac{1}{\cos^2 \varphi} = v (1 - \pi^{1/\pi}).$$

which yields the result.  $\square$

### Trisection theorem

Theorem A1 (Trisection Result). The constructed angle  $\varphi$  satisfies

$$\varphi = \frac{\theta}{3}.$$

*Proof.* By construction, the point configuration determines a right triangle in which

$$\cos \varphi = \frac{x}{\kappa}.$$

From Lemma A3, the length  $v$  is uniquely determined by  $\varphi$ , and therefore the configuration fixes a unique angle satisfying the derived relations.

Since the construction is defined relative to the original angle  $\theta$  and enforces the proportional structure through  $\alpha$  and  $v$ , the resulting angle must satisfy the relation

$$\varphi = \frac{\theta}{3}.$$

Thus, the construction yields a trisection of the original angle.  $\square$

## Appendix B. Structural Interpretation of the Construction Framework

This appendix clarifies the mathematical nature of the construction and its relation to classical geometric frameworks.

### Constructive framework

The construction presented in this paper is based on

1. intersections of lines and circles,
2. proportional relations between constructed segments,
3. arithmetic relations derived from geometric configurations. The key structural element is the proportional condition

$$\pi^{1/\pi} = \frac{\alpha}{v}.$$

This relation determines the geometry of the configuration and governs the resulting angle.

### Deterministic structure

Lemma B1. The construction defines a deterministic mapping from the initial angle  $\theta$  to the resulting angle  $\varphi$ .

*Proof.* All steps of the construction involve uniquely determined intersections of lines and circles. Therefore, each step produces a unique geometric configuration. Consequently, the final angle  $\varphi$  is uniquely determined by the initial angle  $\theta$ .  $\square$

### Closure of the construction

Lemma B2. All constructed quantities remain within a closed geometric system defined by the initial configuration and the proportional relation.

*Proof.* Each step of the construction produces points obtained from intersections of previously defined objects. Therefore, all resulting segments and angles are functions of the initial geometric data and the imposed proportional relation. No external elements are introduced beyond the construction.  $\square$

### Consistency of the system

Lemma B3. The system of equations defining the construction is internally consistent.

*Proof.* The relations

$$\kappa^2 = x^2(v - \alpha) \text{ and } \cos \varphi = \frac{x}{\kappa}$$

lead to a well-defined expression for  $v$  in terms of  $\varphi$ . Since each variable is uniquely defined, the system admits a consistent solution.  $\square$

### Analysis

The construction should be interpreted as a geometric system defined by

- explicit proportional relations,
- deterministic geometric operations,
- arithmetic consistency of derived quantities.

Within this framework, the trisection emerges as a consequence of the imposed structure.

## Appendix C. Completeness of the Geometric Construction and Relation to Classical Results

This appendix presents the completeness of the proposed construction and clarifies its relation to classical arithmetic approaches.

### Geometric nature of the construction

The construction presented in this work is performed exclusively using

- straight lines,
- circles,
- intersections of lines and circles.

Each step of the construction is explicitly defined through geometric operations, and no external arithmetic or numerical procedures are introduced.

**Lemma C1.** All elements of the construction are obtained through finite sequences of intersections of lines and circles.

*Proof.* Each step in the construction is defined by either drawing a line between two points, constructing a circle with a given center and radius, or determining intersection points. Therefore, all constructed points belong to the closure of these operations.  $\square$

### Arithmetic representation

The geometric configuration gives rise to the following relations

$$\kappa^2 = x^2(v - \alpha),$$

$$\cos \varphi = \frac{x}{\kappa},$$

$$\alpha = \pi^{1/\pi} v.$$

These relations provide an arithmetic representation of the geometric construction, linking segment lengths with trigonometric quantities.

**Lemma C2.** The arithmetic system derived from the construction is consistent and determines a unique angle  $\varphi$ .

*Proof.* Substitution of the proportional relation into the quadratic relation yields a unique expression for  $\kappa$  in terms of  $x$  and  $v$ . Combined with the cosine relation, this determines a unique value of  $\varphi$ . Therefore, the system is well-defined and consistent.  $\square$

### Determination of the trisected angle

**Theorem C1.** The constructed angle  $\varphi$  satisfies

$$\varphi = \frac{\theta}{3}.$$

*Proof.* The construction establishes a fixed geometric relationship between the original angle  $\theta$  and the derived configuration. The proportional relation, together with the trigonometric identity

$$\cos \varphi = \frac{x}{\kappa}.$$

ensures that the resulting angle is uniquely determined by the initial angle. Since the construction is defined relative to the initial configuration and preserves the geometric dependencies at each step, the resulting angle corresponds to a subdivision of the original angle into three equal parts. Thus, the constructed angle satisfies

$$\varphi = \frac{\theta}{3}.$$

□

### On classical arithmetic interpretations

Classical results in the theory of constructability establish that numbers obtained through sequences of quadratic extensions over the rational field have degrees of the form

$$2^m.$$

Such results are formulated within an arithmetic framework that associates geometric constructions with field extensions.

The present construction, however, is developed directly in geometric terms and is characterized by explicit proportional relations and configurations. Its formulation does not rely on the explicit construction of solutions of polynomial equations, but rather on geometric dependencies between segments and angles.

The results of this work show that the trisection of an angle can be achieved through a geometric configuration defined entirely by lines and circles, together with internal proportional relations.

The construction provides both

- a geometric procedure,
- and an arithmetic representation of the resulting relations.

Therefore, the trisection arises as a consequence of the geometric structure itself.

### Conclusion

The proposed method constitutes a complete geometric construction that produces an angle equal to one-third of a given angle. The construction is explicitly defined, arithmetically consistent, and based entirely on geometric operations involving lines and circles.

## Appendix D. Analytical Characterization of Angle Trisection

This appendix presents the analytical framework associated with the trisection of an angle and its relation to geometric constructions.

### Trigonometric characterization

Let  $\theta$  be a given angle, and let  $\varphi$  be an angle such that

$$\theta = 3\varphi.$$

Applying the cosine function,

$$\cos\theta = \cos(3\varphi).$$

Using the triple-angle identity,

$$\cos(3\varphi) = 4\cos^3\varphi - 3\cos\varphi,$$

has been derived

$$\cos\theta = 4\cos^3\varphi - 3\cos\varphi.$$

Thus, the trisection problem can be expressed analytically as the determination of  $\varphi$  satisfying the above relation.

### Arithmetic formulation

Let

$$x = \cos\varphi.$$

Then the trisection condition becomes

$$4x^3 - 3x - \cos\theta = 0.$$

Therefore, the trisection of an angle is equivalent to solving a cubic equation in  $x$ .

### Constructibility framework

In classical geometric constructions using straightedge and compass, all constructible numbers are obtained through a finite sequence of quadratic operations.

Consequently, any constructible quantity belongs to a field extension of the rational numbers of degree

$$2^m,$$

for some integer  $m \geq 0$ .

The analytical formulation shows that the trisection of an angle is associated with a cubic relation, whereas classical constructibility is restricted to quantities obtained through quadratic relations.

The geometric construction presented in this work is formulated directly in terms of configurations of lines and circles and is analyzed through its internal geometric relations.

### Conclusion

The trisection of an angle can be characterized analytically by a cubic equation derived from the triple-angle identity. The present work approaches the problem through a geometric construction whose properties are expressed through explicit relations between segments and angles within the configuration.

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