

INVESTIGATING THE FIRST NON-VANISHING COHOMOLOGY OF THE ORLIK-SOLOMON ALGEBRA FOR GRAPHIC ARRANGEMENTS ASSOCIATED WITH FREE-TRIANGLE GRAPHS

Numan Y. Neamah^{1*}, Dr. Hana' M. Ali²

Abstract: In this paper, the first non-vanishing cohomology of the Orlik-Solomon algebra, $A_{-1}^*(G)$, for a graph G having no triangles was investigated where $a = a_{-1} - \lfloor a \rfloor_t$, $1 < t \leq l$, and l is the number of edges in G . Particularly, $H^2(A_{-1}^*(A_G), a)$, did not vanish if G has chordless 4-cycles that contain the edges e_{-1} and $\lfloor e \rfloor_t$.

Introduction

Combinatorics, a branch of Mathematics that organizes and calculates elements under specific constraints, has implications in a variety of fields, including Computer Science, Physics, and Statistics [2, 5]. The matroid serves as one of its essential concepts, since it abstracts the concept of independence in contexts such as Graphs, Networks, and Algebraic structures [15]. Matroids offer a connection between combinatorics and algebraic topology under their association with pure simplicial complexes, which are crucial for the analysis of topological spaces [13]. Hyperplane arrangement A (shortened to “arrangements”), which are collections of affine subspaces of codimension one in a vector space V over a field K , have a topology specified by invariants obtained combinatorically from their intersection lattice $L(A) = \{ \bigcap_{H \in B} H \mid B \subseteq A \}$ [8, 12]. Orlik and Solomon's construction of the Orlik-Solomon algebra $A_{-1}^*(A)$ of an arrangement A [7], calculates the cohomology $H^*(M(A), K)$ of the complement of the arrangement $M(A) = V \setminus \bigcup_{H \in A} H$ and presents techniques for studying cohomology. $H^*(M(A), L(a))$ with the rank one local system's coefficients $L(a)$, $a \in A_{-1}(A)$ [10, 16]. It is possible to generalize $H^*(M(A), K)$ to cohomology with local system $L(a)$, if the non-vanishing cohomology $H^*(A_{-1}^*(A), a)$ of the Orlik-Solomon algebra $A_{-1}^*(A)$ exists [10, 14]. Our study integrates two important fields in mathematics with a wide range of applications, “Arrangements” and “Graphs,” by using matroids for transferring the structure used to construct the Orlik-Solomon algebra $A_{-1}^*(A_G)$ of the graphic arrangement A_G to reconstruct the Orlik-Solomon algebra $A_{-1}^*(G)$ of a graph G using the duality between these two fields [8]. Motivated by our goal, we used the bond lattice $L(G)$ of a triangle-free graph G , to study the first non-

¹ University of Basrah, College of Science, Department of Mathematics.

ORCID ID: <https://orcid.org/0009-0009-2540-5091> Email: nomanzied@gmail.com

² University of Basrah, College of Science, Department of Mathematics.

ORCID ID: <https://orcid.org/0000-0003-2820-0752> Email: hana.ali@uobasrah.edu.iq

*Correspondence: nomanzied@gmail.com

Keywords: Hyperplane arrangement, hypersolvable arrangement, supersolvable arrangement, generic arrangement, Orlik-Solomon algebra, tensor algebra, NBC-module, cohomology of the Orlik-Solomon algebra, graph theory, hypersolvable graph.

vanishing cohomology $H^{**}(A_{-}^{*}(G), a)$ of the Orlik-Solomon algebra $A_{-}^{*}(G)$, where $a = a_{-1} - \sum_{t=1}^l a_t$, $1 < t \leq l$, and l is the number of edges in G . This paper splits into two sections: the first outlines the basic concepts, and the second investigates the cohomology of the Orlik-Solomon algebra for triangle-free graphs. We proved, if G has chordless 4-cycles, then $H^2(A_{-}^{*}(A_G), a)$ is the first non-vanished cohomology of $A_{-}^{*}(A)$.

Basic Concepts

This section is motivated to recall the concepts and structures that we need in our work.

Matroids

Matroid, a crucial concept in combinatorics, encapsulates the idea of independence in various contexts and was first introduced by Hassler Whitney in 1935 [16]. Its foundational structure is a pure simplicial complex, which emerged in the late 19th century via the work of Henri Poincaré [17]. The formal notion of simplicial complexes was later introduced by J. H. C. Whitehead [13].

Definition: [15]

By a finite matroid, we mean a pair $M = (A, \Delta)$, such that Δ is a collection of subsets of a finite set A , satisfying:

1. Δ is an abstract, non-empty simplicial complex, this means, $\Delta \neq \emptyset$ and if $\Delta' \in \Delta$ and $\Delta'' \subset \Delta'$, then $\Delta'' \in \Delta$.
2. All of Δ 's induced subcomplexes are pure, i.e. if $B \subseteq A$, the maximal elements of $\Delta \cap 2^B$ have the same cardinality, where $2^B = \{C \subseteq A \mid C \subseteq B\}$.

Members of Δ are referred to as independent matroid sets, it is thought that the facets represent the matroid's bases, can be write $v \in M$ to mean $v \in A$. It claims that two matroids, $M_1 = (A_1, \Delta_1)$ and $M_2 = (A_2, \Delta_2)$, are isomorphic if and only if $\{\psi(v_1), \dots, \psi(v_k)\} \in \Delta_2$ and a bijection $\psi: A_1 \rightarrow A_2$ such that $\{v_1, \dots, v_k\} \in \Delta_1$. A minimally dependent set is a circuit $C \subseteq A$, i.e. when we remove any o-simplex from C , it becomes independent. Can be define the rank of B by;

$$\text{rk}(B) = \max\{|B'| \mid B' \subseteq B \text{ and } B' \in \Delta\},$$

where $B \subseteq A$. Specifically, $\text{rk}(\emptyset) = 0$ and define the following:

1. The matroid M itself's rank by $\text{rk}(M) = \text{rk}(A) = \dim(\Delta) + 1 = |F|$, where M 's facet is F . Matroid's level is $l(M) = |A| - \text{rk}(M) - 1$.
2. A maximum subset of rank k is called a k -flat of M . Here, it is worth pointing, If a matroid M has flats B and B' , then $B \cap B'$ is also a matroid's flat. The closure $\bar{B}$ of a subset $B \subseteq A$, is the smallest flat containing B , i.e. $\bar{B} = \bigcap_{\text{flats } B' \supseteq B} B'$.
3. The poset of matroid M denoted by $L(M)$, ordered by inclusions. $L(M)$ is a lattice, since it has a top element A , which calls the lattice of flats of M . A unique minimal element $\hat{0} = \emptyset$ has been seen in $L(M)$.
4. Can be define the characteristic polynomial $\chi_M(t)$ of M , as:

$$\chi_M(t) = \sum_{X \in L(M)} \mu(\hat{0}, X) t^{r - \text{rk}(X)};$$

μ denotes to the Möbius function of $L(M)$ and $r = \text{rk}(M)$.

5. The set $\bar{C} = C \setminus v$ is a broken circuit of an ordered matroid $M_{\preceq}$, where v is the minimal element of a circuit C via a total order $\preceq$ on A . The simplicial complex is the definition of the broken circuit complex, also known as the BC-complex.

$$NBC_{\preceq}(M) = \{B \subseteq A \mid B \text{ contains no broken circuit}\}.$$

For $0 \leq k \leq \text{rk}(M)$, set; $NBC_{\preceq}^k(M) = \{B \subseteq A \mid |B| = k, B \text{ contains no broken circuit}\}$. It has been noticed that, if $f^\Delta = (f_0^\Delta, f_1^\Delta, \dots, f_{r-1}^\Delta)$ be the f -vector of $NBC_{\preceq}(M)$, then $|NBC_{\preceq}^k(M)| = f_k^\Delta$ and by applying a result of Rota [11]:

$$\chi_M(t) = f_{-1}^\Delta t^r - f_0^\Delta t^{r-1} + \dots + (-1)^r f_{r-1}^\Delta;$$

where $f_{-1}^\Delta = 1$.

Remark

It is worth noting the following:

1. For a field $K = \mathbb{C}$, let $\mathcal{A}$ be a hyperplane arrangement of a finite dimensional vector space V , we will use $M_{\mathcal{A}} = (\mathcal{A}, \Delta_{\mathcal{A}})$ to denote its related matroid, such that $\Delta_{\mathcal{A}} = \{B \subseteq \mathcal{A} : \text{rk}(B) = |B|\}$, i.e. all of the linearly independent subarrangements of $\mathcal{A}$ are contained in $\Delta_{\mathcal{A}}$, and the lattice of flats $L(M_{\mathcal{A}})$ will be the lattice intersection $L(\mathcal{A})$, for more information, see [4].
2. If $G = (V, \mathcal{E})$ is a simple connected graph such that V and $\mathcal{E}$ be its vertices set and edges set respectively, then $M_G = (\mathcal{E}, \Delta_G)$, will denote its graphic matroid, where Δ_G is the set of all subgraphs that contain no cycle and the lattice of flats $L(M_G)$ will be the lattice of bonds $L(G)$ and its characteristic polynomial $\chi_{M_G}(t)$ represent the chromatic function of G , i.e. $\chi_{M_G}(t)$ is the number of colorings of G with t colors, see [3].
3. The hyperplane arrangements class has a dual subclass within the class of graphs called the graphic arrangements class. Precisely, for every simple connected graph $G = (V, \mathcal{E})$, there is an associated hyperplane arrangement $\mathcal{A}_G$ via the bijection $[v_i, v_j] \in \mathcal{E} \leftrightarrow H_{i,j} \in \mathcal{A}_G$, where $H_{i,j} = \{(x_1, \dots, x_r) \in \mathbb{C}^r : x_i = x_j\}$. Consequently, their matroids M_G and $M_{\mathcal{A}_G}$ are isomorphic. So, their lattices of flats $L(G)$ and $L(\mathcal{A}_G)$ are isomorphic and $\chi_{M_G}(t) = \chi_{M_{\mathcal{A}_G}}(t)$.
4. It is known, the Poincare polynomial of $\mathcal{A}_G$, $P(\mathcal{A}_G, t) = \chi_{M_G}(-t)$. Consequently, if b_j is a j^{th} Betti number of the Poincare polynomial $P(\mathcal{A}_G, t)$, then for $1 \leq j \leq \ell$, $b_j =$ The coloring's number of j vertices of G with t colors.

Definition

By a partition $\Pi = (\Pi_1, \dots, \Pi_l)$ of a matroid $M = (A, \Delta)$ we mean, a partition of A and by a section S of Π , that mean for each $1 \leq k \leq l$, a subset S of A satisfied, either $S \cap \Pi_k$ is empty or a singleton. Let denoted to the set all sections of Π , by $S(\Pi)$, and denoted to the set of all sections of Π with $|S| = k$, by $S_k(\Pi)$. So, can be call the sections of Π , as k -sections. The empty section of Π is a 0-sections. Moreover, can be call Π nice if every choice of l -section of Π is independent. If Π is nice, then $l = \text{rk}(M)$.

Hypersolvable Partition of a Graph

In addition to hypersolvable graphs, our research emphasizes triangle-free graphs. Papadima and Suciu initially introduced hypersolvable graphs in 2002 [9]. The

hypersolvable partition was subsequently devised by Fadhil and Ali as a necessary and sufficient condition for any graph to be considered hypersolvable [6]. We shall utilize the following terminology for defining this concept:

Definition: [16]

The undirected connected simple graph $G=(V, \mathcal{E})$, refer to two finite sets of m -vertices and n -edges respectively, V and $\mathcal{E}$. The hypersolvable partition of G , is a pair $\Pi^G=(\Pi^V, \Pi^{\mathcal{E}})$, and it is denoted by HP , the following facts are satisfy, if $\Pi^V=(\Pi_1^V, \dots, \Pi_{m-1}^V)$ and $\Pi^{\mathcal{E}}=(\Pi_1^{\mathcal{E}}, \dots, \Pi_{\ell}^{\mathcal{E}})$ are partitions of V and $\mathcal{E}$ respectively:

HP₁: $\Pi_1^V = \{v_1, v_2\}$ and $\Pi_1^{\mathcal{E}} = \{e_1\}$, where $e_1 = [v_1, v_2]$, i.e. $\Pi_1^{\mathcal{E}}$ is a singleton.

HP₂: The block Π_j^V is a singleton, for every $2 \leq j \leq m-1$.

HP₃: For every $2 \leq k \leq \ell$, the block $\Pi_k^{\mathcal{E}}$ satisfying the following:

HP_{3i}: For every $e_{i_1}, e_{i_2} \in \Pi_1^{\mathcal{E}} \cup \dots \cup \Pi_k^{\mathcal{E}}$, there is no edge $e \in \Pi_{k+1}^{\mathcal{E}} \cup \dots \cup \Pi_{\ell}^{\mathcal{E}}$ such that $\{e_{i_1}, e_{i_2}, e\}$ forms a triangle.

HP_{3ii}: There is a positive integer $1 < m_k \leq m-1$, such that $V_k = \Pi_1^V \cup \dots \cup \Pi_{m_k}^V$ is a subset of V that contains every endpoints of the edges in $\Pi_1^{\mathcal{E}} \cup \dots \cup \Pi_k^{\mathcal{E}}$, i.e. $G_k = (V_k, \Pi_1^{\mathcal{E}} \cup \dots \cup \Pi_k^{\mathcal{E}})$ makes up a subgraph of G . Then, either.

1. $\Pi_k^{\mathcal{E}} = \{e\}$ such that $V_k = V_{k-1}$, or;
2. $\Pi_k^{\mathcal{E}} = \{e_{i_1}, \dots, e_{i_{d_k}}\}$, such that $V_k \setminus V_{k-1} = \Pi_{m_{k-1}+1}^V = \Pi_{m_k}^V = \{v\}$ and for $1 \leq j \leq d_k$, $e_{i_j} = [v_{i_j}, v]$, for some $v_{i_j} \in \Pi_1^V \cup \dots \cup \Pi_{m_{k-1}}^V$, where $\{v_{i_1}, \dots, v_{i_{d_k}}\} \subseteq V_{k-1} = \Pi_1 \cup \dots \cup \Pi_{m_{k-1}}$ creates a complete subgraph of G .

The number of the blocks of $\Pi^{\mathcal{E}}$ is called the length of Π^G and denoted by $\ell(G) = \ell$. The vector $d = (d_1, \dots, d_{\ell})$ is said to be the exponent vector (or d -vector) of Π if $d_k = |\Pi_k^{\mathcal{E}}|$ for $1 \leq k \leq \ell$. The rank of $\Pi_k^{\mathcal{E}}$ defined as $rk \Pi_k^{\mathcal{E}} = |V_k| - 1$ and $rk(G) = rk \Pi_{\ell}^{\mathcal{E}} = m-1$. The block $\Pi_k^{\mathcal{E}}$ can be considered singular if $|V_{k-1}| = |V_k|$, thus it is non-singular if $|V_k \setminus V_{k-1}| = 1$. Can be consider a hypersolvable partition Π^G is a supersolvable if and only if, $\Pi^{\mathcal{E}}$ has no singular blocks. Also, a hypersolvable partition Π^G , is called m -generic if $\ell \geq m$, the exponent vector $d = (1, \dots, 1)$ and every k -edges of $\mathcal{E}$ cannot be an k -cycle, $3 < k \leq m-1$.

Theorem: [5]

A graph $G = (V, \mathcal{E})$ is said to be supersolvable if and only if there exists a sequence $v_1, v_2, \dots, v_m$ that arranges its vertices so that, for any indices $1 \leq i < j < k \leq m$, if $[v_i, v_k] \in \mathcal{E}$ and $[v_j, v_k] \in \mathcal{E}$, then $[v_i, v_j] \in \mathcal{E}$. The neighborhood of v_i forms a clique when considering only the vertices $v_1, \dots, v_i$ in G .

Proposition: [16]

A graphic arrangement $\mathcal{A}_G$ is hypersolvable (supersolvable or generic), if and only if, it is graph G is hypersolvable (supersolvable or generic).

Note

Let G be a hypersolvable graph. There are important points to take into consideration:

1. The related graphic arrangement $\mathcal{A}_G$ has a hypersolvable partition $\Pi^{\mathcal{A}_G} = (\Pi_1 \dots \Pi_\ell)$ derived from the hypersolvable partition $\Pi^G = (\Pi^\vee, \Pi^\varepsilon)$ on G as; for $1 \leq k \leq \ell$, $H_{ij} \in \Pi_k$ if, and only if, $[i, j] \in \Pi_k^\varepsilon$. We will call $\Pi^{\mathcal{A}_G}$ the induced partition of Π^G [6].
2. According to remark (1.1.2) and definition (1.1.3), there is a bijection between $S(\Pi^{\mathcal{A}_G})$ and the set $S(\Pi^G)$. There is also a bijection between $S_k(\Pi^G)$ and $S_k(\Pi^{\mathcal{A}_G})$, for $1 \leq k \leq \ell$.

Cohomology of OS Algebra

Recalling isomorphic structures of the Orlik-Solomon algebra $\mathbf{A}_*(\mathcal{A})$ is the main goal of this section.

Definition: [13]

Suppose that K is a commutative ring and $\preceq$ is an arbitrary total order defined on the hyperplanes of a ℓ -arrangement $\mathcal{A}$. Define the Orlik-Solomon algebra (or simplicity OS algebra) $\mathbf{A}_*(\mathcal{A})$ to be the quotient of the exterior K -algebra $E_* = \bigwedge_{K \geq 0} (\bigoplus_{H \in \mathcal{A}} K e_H)$, by the homogenous ideal $I_*(\mathcal{A})$ is generated by the relations;

$$\sum_{j=1}^K (-1)^{k-1} e_{H_{i_1}} \dots \hat{e}_{H_{i_j}} \dots e_{H_{i_k}} \text{ for all } 1 \leq i_1 < \dots < i_k \leq n;$$

where $\{H_{i_1}, \dots, H_{i_k}\}$ is a dependent subarrangement of $\mathcal{A}$ and the circumflex “ $\hat{}$ ” indicates the deletion of $e_{H_{i_j}}$.

The broken circuit module $\mathbf{NBC}_*(\mathcal{A})$ is a submodule of the exterior algebra E_* is defined as; $\mathbf{NBC}_0(\mathcal{A}) = K$ and $\mathbf{NBC}_k(\mathcal{A})$ for $1 \leq k \leq \ell$, be the free K -module of E_k with an NBC (no broken circuit) monomial basis $\{e_C | C \in \mathbf{NBC}_k(\mathcal{A})\} \subseteq E_k$, i.e.;

$$\mathbf{tNBC}_k(\mathcal{A}) = \bigoplus_{C \in \mathbf{NBC}_k(\mathcal{A})} K e_C \text{ and } \mathbf{tNBC}_*(\mathcal{A}) = \bigoplus_{k=0}^{\ell} \mathbf{NBC}_k(\mathcal{A}).$$

Theorem: [13]

On the broken circuit submodule $\mathbf{NBC}_*(\mathcal{A})$, the restriction of canonical chain map $\varphi_*: E_* \rightarrow \mathbf{A}_*(\mathcal{A})$, is an isomorphism defined as; for $1 \leq k \leq \ell$, $\varphi_k(e_C) = e_C + I_k(\mathcal{A}) = a_C$, $C \in \mathbf{NBC}_k(\mathcal{A})$. Consequently, the OS algebra $\mathbf{A}_*(\mathcal{A})$ embedded by the following structure as a free K -submodule of the exterior algebra: $\mathbf{A}_*(\mathcal{A}) = \bigoplus_{k=0}^{\ell} (\bigoplus_{C \in \mathbf{NBC}_k(\mathcal{A})} K a_C)$.

Definition: [9]

Assume that $a \in \mathbf{A}_1(\mathcal{A})$ with $a = \sum_{s=1}^n \lambda_s a_s$ for $\lambda_s \in K$. multiplication by a give a differentiation $d_k: \mathbf{A}_k(\mathcal{A}) \xrightarrow{a} \mathbf{A}_{k+1}(\mathcal{A})$ such that $(\mathbf{A}_*(\mathcal{A}), a)$ forms a complex. The cohomology of this complex denoted by $H^*(\mathbf{A}(\mathcal{A}), a)$ and is said to be the cohomology of the Orlik-Solomon algebra.

Main Results

Our work involves two parts. First, we reconstruct the Orlik-Solomon algebra for a graph by utilizing its hypersolvable partition structure. Second, we examine the first non-vanishing cohomology for a triangle-free graph.

The OS Algebra of a Graph

Here, we use the hypersolvable partition structure of a triangle-free graph to constrain the cohomology structure of the Orlik-Solomon algebra. We review the Orlik-Solomon algebra's structure as offered in [1] in order to make our methodology clearer:

Construction

Let Q_m be a complete graph with m vertices and assume $\leq_{Q_m}^V$ is a total order defined on the set of its vertices say as; $V_{Q_m} = \{v_1, \dots, v_m\}$. Via $\leq_{Q_m}^V$ we will define a total order on its edges that is induced from its supersolvability. We conduct this by recalling the strong geometrical features of its supersolvable partition on $\mathcal{E}_{Q_m}$:

$$\Pi^{\mathcal{E}_{Q_m}} = \left(\Pi_1^{\mathcal{E}_{Q_m}}, \Pi_2^{\mathcal{E}_{Q_m}}, \Pi_3^{\mathcal{E}_{Q_m}}, \dots, \Pi_{m-1}^{\mathcal{E}_{Q_m}} \right)$$

$$= t(\{[v_1, v_2]\}, \{[v_1, v_3], [v_2, v_3]\}, t\{[v_1, v_4], [v_2, v_4], [v_3, v_4]\}, \dots, \{[v_1, v_m], [v_2, v_m], \dots, [v_{m-1}, v_m]\})$$

to define an order $\leq_{Q_m}^{\mathcal{E}}$ on $\mathcal{E}_{Q_m}$ that respect this partition as follows:

$$\begin{aligned} [v_1, v_2] \leq_{Q_m}^{\mathcal{E}} [v_1, v_3] \leq_{Q_m}^{\mathcal{E}} [v_2, v_3] \leq_{Q_m}^{\mathcal{E}} [v_1, v_4] \leq_{Q_m}^{\mathcal{E}} [v_2, v_4] \leq_{Q_m}^{\mathcal{E}} [v_3, v_4] \leq_{Q_m}^{\mathcal{E}} \dots \\ \leq_{Q_m}^{\mathcal{E}} [v_1, v_m] \leq_{Q_m}^{\mathcal{E}} [v_2, v_m] \leq_{Q_m}^{\mathcal{E}} \dots \leq_{Q_m}^{\mathcal{E}} [v_{m-1}, v_m]. \end{aligned}$$

Straight to the definition of $\leq_{Q_m} = (\leq_{Q_m}^V, \leq_{Q_m}^{\mathcal{E}})$, we have the following:

1. If $i < j$, $[v_{k_i}, v_{i+1}] \in \Pi_i^{\mathcal{E}_{Q_m}}$ and $[v_{k_j}, v_{j+1}] \in \Pi_j^{\mathcal{E}_{Q_m}}$, then $[v_{k_i}, v_{i+1}] \leq_{Q_m}^{\mathcal{E}} [v_{k_j}, v_{j+1}]$, for each $1 \leq k_i \leq i$ and $1 \leq k_j \leq j$.
2. If $1 \leq k_1 < k_2 < k_3 \leq k$, then in block $\Pi_k^{\mathcal{E}_{Q_m}}$ we have:
 - ✓ $[v_{k_1}, v_{k+1}] \leq_{Q_m}^{\mathcal{E}} [v_{k_2}, v_{k+1}] \leq_{Q_m}^{\mathcal{E}} [v_{k_3}, v_{k+1}]$ and;
 - ✓ $[v_{k_1}, v_{k_2}] \leq_{Q_m}^{\mathcal{E}} [v_{k_1}, v_{k_3}] \leq_{Q_m}^{\mathcal{E}} [v_{k_2}, v_{k_3}]$.

We will employ the above structure of $\leq_{Q_m}$ on the complete graph Q_m to establish an order $\leq_G$ defined on any hypersolvable graph $G = (V, \mathcal{E})$ with m vertices through its HP $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$ via the inclusion map $i_G: G \hookrightarrow Q_m$, which we designate as hypersolvable order. That is, the order $\leq_G$ will satisfy:

1. If $v_i \in \Pi_i^V$ and $v_j \in \Pi_j^V$ such that; $i < j$, then $v_i \leq_G^V v_j$.
2. If $e \in \Pi_i^{\mathcal{E}}$ and $e' \in \Pi_j^{\mathcal{E}}$ such that; $i < j$, then $e \leq_G^{\mathcal{E}} e'$.
3. If $e_{i_1}, e_{i_2}, e_{i_3} \in \Pi_k^{\mathcal{E}}$ and $e_{i_1} \leq_G^{\mathcal{E}} e_{i_2} \leq_G^{\mathcal{E}} e_{i_3}$, then $e_{i_1, i_2} \leq_G^{\mathcal{E}} e_{i_1, i_3} \leq_G^{\mathcal{E}} e_{i_2, i_3}$, where $\{e_{i_1, i_2}, e_{i_1}, e_{i_2}\}, \{e_{i_1, i_3}, e_{i_1}, e_{i_3}\}, \{e_{i_2, i_3}, e_{i_2}, e_{i_3}\}$ are the triangles in G via the solvable property of Π^G .

Construction

From this point on, we assume that $G = (V, \mathcal{E})$ be a simple connected triangle-free graph such that the number of its vertices is $m \geq 4$, the number of its edges is ℓ and $M_G = (\mathcal{E}, \Delta_G)$ represent its Matroid. Therefore, G is hypersolvable with an HP say $\Pi^G = (\Pi^V, \Pi^{\mathcal{E}})$. Consequently, G has an exponent vector $w = (1, \dots, 1)$ such that the length of Π^G is $\ell(G) = \ell = |\mathcal{E}| = |\mathcal{A}_G|$.

In construction (2.1.1), it is crucial to recollect the hypersolvable order $\leq_G$ on G to establish an order $\leq_{\mathcal{A}_G}$ on the hyperplanes of $\mathcal{A}_G$, based on the structure of the

induced partition $\Pi^{\mathcal{A}_G}$ as outlined in note (1.2.4). Consequently, due to remark (1.1.2) and the isomorphism between the lattice of bonds $L(G)$ and the lattice of intersections $L(\mathcal{A}_G)$, we shall derive the following bijections via the hypersolvable order $\preceq_G$:

1. $f: \text{NBC}_{\preceq_G}(M_G) \rightarrow \text{NBC}_{\preceq_{\mathcal{A}_G}}(M_{\mathcal{A}_G})$.
2. For $0 \leq k \leq \text{rk}(M_G)$, $f_k: \text{NBC}_{\preceq_G}^k(M_G) \rightarrow \text{NBC}_{\preceq_{\mathcal{A}_G}}^k(M_{\mathcal{A}_G})$ will be the restriction of f on the k^{th} - skeleton of $\text{NBC}_{\preceq_G}^k(M_G)$.
3. $g: S_{\preceq_G}(\Pi^G) \rightarrow S_{\preceq_{\mathcal{A}_G}}(\Pi^{\mathcal{A}_G})$.
4. For $0 \leq k \leq \ell(G)$, $g_k: S_{\preceq_G}^k(\Pi^G) \rightarrow S_{\preceq_{\mathcal{A}_G}}^k(\Pi^{\mathcal{A}_G})$ will be the restriction of g on $S_{\preceq_G}^k(\Pi^G)$.

According [1], we have,

$$p(M_G) = \text{Max}\{k \mid |\text{NBC}_{\preceq_G}^k(M_G)| = |S_{\preceq_G}^k(\Pi^G)|\} = c - 2;$$

where $c = c(G) = \text{Min}\{|C| : C \text{ is a } j - \text{cycle with no chord, } j \geq 4\}$. As a result, we can categorize the class of triangle-free graphs into three subclasses, and we can reconstruct the Orlik-Solomon algebra as free K -module and will be denoted by $\mathbf{A}_*(G)$ using the following formula:

If $\ell(G) = m - 1$, then G is a tree which is supersolvable, and the Orlik-Solomon algebra reconstructed as:

$$\mathbf{A}_*(G) \cong \bigoplus_{k=0}^{m-1} \left(\bigoplus_{C \in S_{\preceq_G}^k(\Pi^G)} \text{Ka}_C \right) \text{ and for } 1 \leq k \leq m - 1, \mathbf{A}_k(G) \cong \bigoplus_{C \in S_{\preceq_G}^k(\Pi^G)} \text{Ka}_C \\ \text{and, } b_k(\mathbf{A}_*(G)) = \binom{m-1}{k}.$$

If $c(G) = \ell(G) = m$, then G is generic that form's m -cycle with no chord, and the Orlik-Solomon algebra reconstructed as: for $1 \leq k \leq m - 2$, $\mathbf{A}_k(G) \cong \bigoplus_{C \in S_{\preceq_G}^k(\Pi^G)} \text{Ka}_C$ with $b_k(\mathbf{A}_*(G)) = \binom{m}{k}$ and, $\mathbf{A}_{m-1}(G) \cong \bigoplus_{C \in \text{NBC}_{\preceq_G}^{m-1}(M_G)} \text{Ka}_C$, where $\text{NBC}_{\preceq_G}^{m-1}(M_G) = S_{\preceq_G}^{m-1}(\Pi^G) - \{\mathcal{E} - \{[v_1, v_2]\}\}$ and, $b_{m-1}(\mathbf{A}_*(G)) = m - 1$. Thus,

$$\mathbf{A}_*(G) \cong \bigoplus_{k=0}^{m-2} \left(\bigoplus_{C \in S_{\preceq_G}^k(\Pi^G)} \text{Ka}_C \right) \oplus \left(\bigoplus_{C \in S_{\preceq_G}^{m-1}(\Pi^G) - \{\mathcal{E} - \{[v_1, v_2]\}\}} \text{Ka}_C \right).$$

If $c(G) = c \leq m - 1 < \ell$, then G is neither supersolvable nor generic and for $1 \leq k \leq c - 2$, $\mathbf{A}_k(G) \cong \bigoplus_{C \in S_{\preceq_G}^k(\Pi^G)} \text{Ka}_C$ with $b_j(\mathbf{A}_*(G)) = |\text{NBC}_k(\mathcal{A}_G)| = \binom{m}{k}$ and $\mathbf{A}_{c-1}(G) \cong \bigoplus_{C \in \text{NBC}_{\preceq_G}^{c-1}(M_G)} \text{Ka}_C$, where $\text{NBC}_{\preceq_G}^{c-1}(M_G) = S_{\preceq_G}^{c-1}(\Pi^G) - \text{BC}_{\preceq_G}^{c-1}(M_G)$, $\text{BC}_{\preceq_G}^{c-1}(M_G)$ is the set of all broken c -cycles via $\preceq_G^{\mathcal{E}}$, and $b_{c-1}(\mathbf{A}_*(G)) = \binom{m}{c-1} - |\text{BC}_{\preceq_G}^{c-1}(M_G)|$. Thus:

$$\mathbf{A}_*(G) \cong \bigoplus_{k=0}^{m-1} \left(\bigoplus_{C \in \text{NBC}_{\preceq_G}^k(M_G)} \text{Ka}_C \right) \\ \cong \bigoplus_{k=0}^{c-2} \left(\bigoplus_{C \in S_{\preceq_G}^k(\Pi^G)} \text{Ka}_C \right) \\ \oplus \left(\bigoplus_{C \in S_{\preceq_G}^{c-1}(\Pi^G) - \text{BC}_{\preceq_G}^{c-1}(M_G)} \text{Ka}_C \right) \oplus \bigoplus_{k=c}^{m-1} \left(\bigoplus_{C \in \text{NBC}_{\preceq_G}^k(M_G)} \text{Ka}_C \right)$$

The Structure of $H^2(\mathbf{A}_*(G), \mathbf{a})$

From now on, assume G is a triangle-free graph and we reordered its set of edges $\mathcal{E} = \{e_1, e_2, \dots, e_\ell\}$ of G by the hypersolvable order $\preceq_G$, i.e. $e_1 \preceq_G^{\mathcal{E}} e_2 \preceq_G^{\mathcal{E}} \dots \preceq_G^{\mathcal{E}} e_\ell$.

Furthermore, for a fixed $a \in \mathbf{A}_1(G) \cong \bigoplus_{e_t \in E} Ka_{e_t}$, recall the complex $(\mathbf{A}_*(G), a)$ due definition (1.3.6) for constructing the cohomology of the Orlik-Solomon algebra $H^*(\mathbf{A}_*(G), a)$.

Lemma

If $a = a_{e_1} - a_{e_t}$, for $2 \leq t \leq \ell$, then $\dim(\text{Im } d_1) = \ell - 1$;

Proof

Refer to construction (2.1.2), specifically, $\mathbf{A}_2(G) \cong \bigoplus_{C \in S_{\leq G}^2} Ka_C$. To achieve our aim, we will analyze the homomorphism, $d_1: \mathbf{A}_1(G) \xrightarrow{a} \mathbf{A}_2(G)$. It is significant to highlight that the basis for $\mathbf{A}_2(G)$ is $\{a_C: C \in S_{\leq G}^2(\Pi^G)\}$ which includes all the monomials of the 2-sections of Π^E . Thus, for $1 \leq s \leq \ell$, we have:

$$d_1(a_{e_s}) = a_{e_s}(a_{e_1} - a_{e_t}) = \begin{cases} -a_{e_1}a_{e_t} & : s = 1 \text{ or } t \\ -a_{e_1}a_{e_s} - a_{e_s}a_{e_t} & : 1 < s < t \leq \ell \\ -a_{e_1}a_{e_s} + a_{e_t}a_{e_s} & : 1 < t < s \leq \ell \end{cases}$$

Indeed G is hypersolvable, hence every 2-section of Π^E is an NBC base of Δ_G . Therefore, $d_1(a_{e_s}) \neq 0_{\mathbf{A}_2(G)}$, since it is expressed as a combination of NBC monomials. Moreover, considering that our graph G is devoid of triangles, it implies that each 3-section of Π^E is either an NBC base or broken circuit of Δ_G . So, for $1 < s < t \leq \ell$,

$$\begin{aligned} \partial_3^{\mathbf{A}_*(G)}(a_{e_1}a_{e_s}a_{e_t}) &= a_{e_s}a_{e_t} - a_{e_1}a_{e_t} + a_{e_1}a_{e_s} = -d_1(a_{e_s}) - a_{e_1}a_{e_t} \neq 0_{\mathbf{A}_2(G)} \\ &\Rightarrow d_1(a_{e_s}) \neq -a_{e_1}a_{e_t} \end{aligned}$$

As well as, for $1 < t < s \leq \ell$,

$$\begin{aligned} \partial_3^{\mathbf{A}_*(G)}(a_{e_1}a_{e_t}a_{e_s}) &= a_{e_t}a_{e_s} - a_{e_1}a_{e_s} + a_{e_1}a_{e_t} = d_1(a_{e_s}) + a_{e_1}a_{e_t} \neq 0_{\mathbf{A}_2(G)} \\ &\Rightarrow d_1(a_{e_s}) \neq -a_{e_1}a_{e_t} \end{aligned}$$

Therefore, the basis of $\text{Im } d_1$ included $\ell - 1$ monomials and $\dim(\text{Im } d_1) = \ell - 1$.

Proposition

If $a = a_{e_1} - a_{e_t}$, for $2 \leq t \leq \ell$, then $\ker d_2 = \text{Im } d_1$ and;

1. If G is tree or $c(G) > 4$, then $\dim(\text{Im } d_2) = \binom{\ell - 2}{2} + (\ell - 2)$.
2. If $c(G) = 4$, then $\dim(\text{Im } d_2) = \binom{\ell - 2}{2} + (\ell - 2) - u_4(G)$, such that $u_4(G)$ be the number of chordless 4-cycles that includes e_1 and e_t .

Proof

Our objective will drive us to study the homomorphism, $d_2: \mathbf{A}_2(G) \xrightarrow{a} \mathbf{A}_3(G)$. Due construction (2.1.2), the structure of $\mathbf{A}_3(G)$ depend on the value of $c(G) = \text{Min}\{|C|: C \text{ is a } j - \text{cycle with no chord, } j \geq 4\}$. As previously mentioned, $m \geq 4$, hence either (G is tree or $c(G) > 4$) or ($c(G) = 4$). So, either:

$$\begin{aligned} (\text{NBC}_{\leq G}^3(M_G) = S_{\leq G}^3(\Pi^G) \text{ and } \mathbf{A}_3(G) \cong \bigoplus_{c \in S_{\leq G}^3(\Pi^G)} \text{Ka}_c, \text{ or;} \\ (\text{NBC}_{\leq G}^3(M_G) = S_{\leq G}^3(\Pi^G) - \text{BC}_{\leq G}^3(M_G) \text{ and } \mathbf{A}_3(G) \cong \bigoplus_{c \in S_{\leq G}^3(\Pi^G) - \text{BC}_{\leq G}^3(M_G)} \text{Ka}_c; \end{aligned}$$

where $\text{BC}_{\leq G}^3(M_G)$ obtained by removing the minimal edge via $\leq_G$ from the 4-cycles with no chord. So, the basis for $\mathbf{A}_3(G)$ depend on $c(G)$. In general, for $1 \leq k_1 < k_2 \leq \ell$, we have;

$$d_2(a_{e_{k_1}} a_{e_{k_2}}) = a_{e_{k_1}} a_{e_{k_2}} (a_{e_1} - a_{e_t})$$

$$= \begin{cases} 0_{\mathbf{A}_3(G)} & : k_1 = 1 \text{ and } k_2 = t \\ -a_{e_1} a_{e_{k_2}} a_{e_t} & : 1 = k_1 < k_2 < t \leq \ell \\ a_{e_1} a_{e_t} a_{e_{k_2}} & : 1 = k_1 < t < k_2 \leq \ell \\ a_{e_1} a_{e_{k_1}} a_{e_t} & : 1 < k_1 < k_2 = t \leq \ell \\ a_{e_1} a_{e_t} a_{e_{k_2}} & : 1 < k_1 = t < k_2 \leq \ell \\ a_{e_1} a_{e_{k_1}} a_{e_{k_2}} - a_{e_{k_1}} a_{e_{k_2}} a_{e_t} & : 1 < k_1 < k_2 < t \leq \ell \\ a_{e_1} a_{e_{k_1}} a_{e_{k_2}} + a_{e_{k_1}} a_{e_t} a_{e_{k_2}} & : 1 < k_1 < t < k_2 \leq \ell \\ a_{e_1} a_{e_{k_1}} a_{e_{k_2}} - a_{e_t} a_{e_{k_1}} a_{e_{k_2}} & : 1 < t < k_1 < k_2 \leq \ell \end{cases} \quad \dots (2.2.2.1)$$

It is clear, for $1 < k < t$ or $t < k \leq \ell$, $d_2(a_{e_1} a_{e_k}) = -d_2(a_{e_k} a_{e_t}) \neq 0_{\mathbf{A}_3(G)}$, since they represent NBC-monomials. Based on the configuration of G , we show the following:

1. If G is tree or $c(G) > 4$, we have $\text{NBC}_{\leq G}^3(M_G) = S_{\leq G}^3(\Pi^G)$. Formula (2.2.2.1) states that, $d_2(a_{e_{k_1}} a_{e_{k_2}}) = 0_{\mathbf{A}_3(G)}$ for $k_1 = 1$ and $k_2 = t$, and for the other cases it constitutes a combination of NBC-monomials. Thus, $a_{e_1} a_{e_t} \in \ker d_2$. And since, $(A_*(\mathcal{A}), a)$ is a complex, therefore, $\text{Im } d_1 = \ker d_2$ and $\dim(\text{Im } d_2) = \binom{\ell-2}{2} + (\ell-2)$.
2. If $c(G) = 4$, then there are two possibilities:

If $m = 4$, then G is 4-generic has 4 edges, say $\mathcal{E} = \{e_1 = [v_1, v_2], e_2 = [v_2, v_3], e_3 = [v_1, v_4], e_4 = [v_3, v_4]\}$. Since, $\partial_4^{A_*(G)}(a_{\mathcal{E}}) = \partial_4^{A_*(G)}(a_{e_1} a_{e_2} a_{e_3} a_{e_4}) = 0_{\mathbf{A}_3(G)}$, hence;

$$a_{e_2} a_{e_3} a_{e_4} = a_{e_1} a_{e_3} a_{e_4} - a_{e_1} a_{e_2} a_{e_4} + a_{e_1} a_{e_2} a_{e_3} \dots (2.2.2.2)$$

Thus, for $1 \leq k_1 < k_2 \leq 4$, below are three cases for values of t :

If $t = 2$, then:

$$\begin{aligned} d_2(a_{e_1} a_{e_2}) &= 0_{\mathbf{A}_3(G)}, d_2(a_{e_1} a_{e_3}) = d_2(a_{e_2} a_{e_3}) = a_{e_1} a_{e_2} a_{e_3}, \\ d_2(a_{e_1} a_{e_4}) &= d_2(a_{e_2} a_{e_4}) = a_{e_1} a_{e_2} a_{e_4}, d_2(a_{e_3} a_{e_4}) = -d_2(a_{e_1} a_{e_3}) + d_2(a_{e_2} a_{e_3}). \end{aligned}$$

If $t = 3$, then:

$$\begin{aligned} d_2(a_{e_1} a_{e_2}) &= -d_2(a_{e_2} a_{e_3}) = -a_{e_1} a_{e_2} a_{e_3}, d_2(a_{e_1} a_{e_3}) = -0_{\mathbf{A}_3(G)}, d_2(a_{e_1} a_{e_4}) = \\ d_2(a_{e_3} a_{e_4}) &= a_{e_1} a_{e_3} a_{e_4} \text{ and } d_2(a_{e_2} a_{e_4}) = -d_2(a_{e_1} a_{e_2}) + d_2(a_{e_1} a_{e_4}). \end{aligned}$$

If $t = 4$, then:

$$\begin{aligned} d_2(a_{e_1} a_{e_2}) &= d_2(a_{e_2} a_{e_4}) = -a_{e_1} a_{e_2} a_{e_4}, d_2(a_{e_1} a_{e_3}) = d_2(a_{e_3} a_{e_4}) = -a_{e_1} a_{e_3} a_{e_4}, \\ d_2(a_{e_1} a_{e_4}) &= 0_{\mathbf{A}_3(G)} \text{ and } d_2(a_{e_2} a_{e_3}) = d_2(a_{e_2} a_{e_4}) = -d_2(a_{e_1} a_{e_2}) + d_2(a_{e_1} a_{e_3}). \end{aligned}$$

Therefore, $a_{e_1} a_{e_t} \in \ker d_2$ and $\dim(\text{Im } d_2) = \binom{4-2}{2} + (4-2) - 1 = 2$. As a direct result of the structure of the complex $(\mathbf{A}_*(\mathcal{A}), a)$, $\ker d_2 = \text{Im } d_1$.

Let c_4 be the number of all chordless 4-cycles of G and assume that we have $u_4 \geq 1$ of chordless 4-cycles that contain e_1 and e_t , if $m > 4$. Let C be a chordless 4-cycle characterized by $\{e_{q_1}, e_{q_2}, e_{q_3}, e_{q_4}\}$. This means that its broken cycle monomial can be expressed as:

$$a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_4}} = a_{e_{q_1}} a_{e_{q_3}} a_{e_{q_4}} - a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_4}} + a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_3}} \dots (2.2.2.3)$$

Therefore, we have the following possible cases:

The following possibilities apply to $(c_4 - u_4)$ of chordless 4-cycles that do not simultaneously include e_1 and e_t :

b.1.1. If $e_1, e_t \notin C$, then formula (2.2.2.1) shows that $d_2(a_{q_i} a_{q_j})$ cannot include

$a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_4}}$, for every $1 \leq i < j \leq 4$. So, $d_2(a_{q_i} a_{q_j})$ written as a combination of NBC-monomials.

b.1.2. The second case here is: either $(e_1 \notin C \text{ and } t \in C)$ or $(e_1 \in C \text{ and } e_t \notin C)$. If $e_1 \notin C$ and there exists $1 \leq j \leq 4$ such $e_{q_j} = e_t$. Thus, for $1 \leq i < k \leq 4$ and $i, k \neq t$, we get either $d_2(a_{e_1} a_{e_{q_i}}) = \mp d_2(a_{e_{q_i}} a_{e_t}) = \mp a_{e_1} a_{e_{q_i}} a_{e_t}$ or $d_2(a_{e_{q_i}} a_{e_{q_k}}) = a_{e_1} a_{e_{q_i}} a_{e_{q_k}} \mp a_{e_{q_i}} a_{e_{q_k}} a_{e_t}$. It is clear that $a_{e_1} a_{e_{q_i}} a_{e_t}$ is an NBC monomial. For $d_2(a_{e_{q_i}} a_{e_{q_j}})$ and $i, k \neq t$, deduce that either it is written as a combination of NBC monomial if $a_{e_s} a_{e_{q_i}} a_{e_{q_j}} \neq \mp a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_4}}$ or if $a_{e_s} a_{e_{q_i}} a_{e_{q_j}} = \mp a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_4}}$, so by substituting formula (2.2.2.3), $d_2(a_{e_{q_i}} a_{e_{q_j}}) = a_{e_1} a_{e_{q_i}} a_{e_{q_k}} \mp (a_{e_{q_1}} a_{e_{q_3}} a_{e_{q_4}} - a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_4}} + a_{e_{q_1}} a_{e_{q_2}} a_{e_{q_3}})$, will be written as a combination of NBC-monomial. The number of the generators combining the basis of $\text{Im } d_2$ is 3, calculated as $\binom{4-2}{2} + (4-2) = 3$. Similarly, we can deduce that, in the case where $(e_t \notin C \text{ and } e_1 \in C)$, three generators included in $\text{Im } d_2$.

Assume C is one of the u_4 chordless 4-cycles, that including e_1 and e_t , for the purpose of seeing how one of the 2-NBC monomial images cannot be including in $\text{Im } d_2$'s basis, as follows:

b.2.1. If $t = q_2$: $d_2(a_{e_{q_3}} a_{e_{q_4}}) = d_2(a_{e_{q_1}} a_{e_{q_3}}) - d_2(a_{e_{q_1}} a_{e_{q_4}})$.

b.2.2. If $t = q_3$: $d_2(a_{e_{q_2}} a_{e_{q_4}}) = d_2(a_{e_{q_1}} a_{e_{q_4}}) - d_2(a_{e_{q_1}} a_{e_{q_3}})$.

b.2.3. If $t = q_4$: $d_2(a_{e_{q_2}} a_{e_{q_3}}) = d_2(a_{e_{q_1}} a_{e_{q_3}}) - d_2(a_{e_{q_1}} a_{e_{q_2}})$.

In this case, the generators including the basis of $\text{Im } d_2$, are exactly two elements calculated as: $\binom{4-2}{2} + (4-2) - 1 = 2$.

b.1. The number of generators entering the basis of $\text{Im } d_2$'s basis is $\binom{6-2}{2} + (6-2) - 2 = 8$, if two chordless 4-cycles intersected with e_1 and e_t .

- b.2.** If two chordless 4-cycles intersected with e_1 or e_t , the number of generators combining $\text{Im } d_2$'s basis is either $\binom{7-2}{2} + (7-2) = 15$ or $\binom{7-2}{2} + (7-2) - 1 = 14$, depending on whether one of them contains e_1 and e_t .

As result, $d_2(a_{e_{k_1}} a_{e_{k_2}}) = 0_{\mathbf{A}_3(G)}$, if and only if $k_1 = 1$ and $k_2 = t$, that implies $\ker d_2 = \text{Im } d_1$. Therefore, $\dim(\text{Im } d_2) = \binom{\ell-2}{2} + (\ell-2) - u_4$.

Theorem

If $a = a_{e_1} - a_{e_t}$, for $2 \leq t \leq \ell$, then:

1. If G is tree $\text{orc}(G) > 5$, then $\ker d_3 = \text{Im } d_2$ and $\dim(\text{Im } d_3) = \binom{\ell-2}{3} + \binom{\ell-2}{2}$.
2. If $c(G) = 5$, then $\ker d_3 = \text{Im } d_2$ and $\dim(\text{Im } d_3) = \binom{\ell-2}{3} + \binom{\ell-2}{2} - u_5(G)$.
3. If G is 4-generic graph, then $\ker d_3 = \mathbf{A}_3(G)$.
4. If $c(G) = 4$, then $\dim(\ker d_3) = \dim(\text{Im } d_2) + u_4$ and:

$$\dim(\text{Im } d_3) = \left(\binom{\ell-2}{3} - (u_4(\ell-4) - b_4) - (c_4 - u_4) \right) + \left(\binom{\ell-2}{2} - (u_4 + v_4) \right) - u_5.$$

where c_4 be the number of chordless 4-cycles, u_4 and u_5 be the number of chordless 4-cycles and chordless 5-cycles that includes e_1 and e_t respectively, v_4 is the number of chordless 4-cycles that includes e_t as not minimal edge via $\trianglelefteq_G$ and b_4 is the number of broken circuits that contains $B - \{e_1, e_t\}$, and B is a chordless 4-cycle that simultaneously include e_1 and e_t .

Proof

According to construction (2.1.2), the structure of $\mathbf{A}_4(G)$ depends on the value of $c(G)$. As we mentioned, the graph G has no triangles, hence either (G is tree or $c(G) > 4$) or ($c(G) = 4$). So:

1. If G is tree or $c(G) > 5$, then $\text{NBC}_{\trianglelefteq_G}^4(M_G) = S_{\trianglelefteq_G}^4(\Pi^G)$ and $\mathbf{A}_4(G) \cong \bigoplus_{C \in S_{\trianglelefteq_G}^4(\Pi^G)} \text{Ka}_C$.
2. If $c(G) = 5$, then $\text{NBC}_{\trianglelefteq_G}^4(M_G) = S_{\trianglelefteq_G}^4(\Pi^G) - \text{BC}_{\trianglelefteq_G}^4(M_G)$ and;

$$\mathbf{A}_4(G) \cong \bigoplus_{C \in S_{\trianglelefteq_G}^4(\Pi^G) - \text{BC}_{\trianglelefteq_G}^4(M_G)} \text{Ka}_C,$$

3. If $c(G) = 4$, then $\text{NBC}_{\trianglelefteq_G}^4(M_G) = S_{\trianglelefteq_G}^4(\Pi^G) - (\text{BC}_{\trianglelefteq_G}^4(M_G) \cup C_{\trianglelefteq_G}^4(M_G))$ and;

$$\mathbf{A}_4(G) \cong \bigoplus_{C \in S_{\trianglelefteq_G}^4(\Pi^G) - (\text{BC}_{\trianglelefteq_G}^4(M_G) \cup C_{\trianglelefteq_G}^4(M_G))} \text{Ka}_C,$$

Where, where $\text{BC}_{\trianglelefteq_G}^4(M_G)$ contains the sections that obtained by deleting the minimal edge via $\trianglelefteq_G$ from the chordless 5-cycles and $C_{\trianglelefteq_G}^4(M_G)$ include the sections that related to chordless 4-cycles ordered via $\trianglelefteq_G$.

Now, we need to examine the homomorphism, $d_3: \mathbf{A}_3(G) \xrightarrow{a} \mathbf{A}_4(G)$, according to the value of $c(G)$. In general, for $1 \leq k_1 < k_2 < k_3 \leq \ell$,

$$\begin{aligned}
d_3(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}) &= a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} (a_{e_1} - a_{e_t}) \\
&= \begin{cases} 0_{A_4(G)} & : k_i = 1 \text{ and } k_j = t \text{ for some } 1 \leq i < j \leq 3 \\
\begin{aligned}
&-a_{e_1} a_{e_{k_2}} a_{e_{k_3}} a_{e_t} & : 1 = k_1 < k_2 < k_3 < t \leq \ell \\
&a_{e_1} a_{e_{k_2}} a_{e_t} a_{e_{k_3}} & : 1 = k_1 < k_2 < t < k_3 \leq \ell \\
&-a_{e_1} a_{e_t} a_{e_{k_2}} a_{e_{k_3}} & : 1 = k_1 < t < k_2 < k_3 \leq \ell \\
&-a_{e_1} a_{e_{k_1}} a_{e_{k_2}} a_{e_t} & : 1 < k_1 < k_2 < k_3 = t \leq \ell \\
&a_{e_1} a_{e_{k_1}} a_{e_t} a_{e_{k_3}} & : 1 < k_1 < k_2 = t < k_3 \leq \ell \\
&-a_{e_1} a_{e_t} a_{e_{k_2}} a_{e_{k_3}} & : 1 < k_1 = t < k_2 < k_3 \leq \ell \\
&a_{e_1} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} - a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} a_{e_t} & : 1 < k_1 < k_2 < k_3 < t \leq \ell \\
&a_{e_1} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} + a_{e_{k_1}} a_{e_{k_2}} a_{e_t} a_{e_{k_3}} & : 1 < k_1 < k_2 < t < k_3 \leq \ell \\
&a_{e_1} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} - a_{e_{k_1}} a_{e_t} a_{e_{k_2}} a_{e_{k_3}} & : 1 < k_1 < t < k_2 < k_3 \leq \ell \\
&a_{e_1} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} + a_{e_t} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}} & : 1 < t < k_1 < k_2 < k_3 \leq \ell
\end{aligned} & \quad (2.2.3.1)
\end{cases}
\end{aligned}$$

Within the type G , we will look at all possible cases that follow:

1. The relation $NBC_{\leq G}^4(M_G) = S_{\leq G}^4(\Pi^G)$ holds if G is a tree or $c(G) > 5$. If $k_i = 1$ and $k_j = t$ for some $1 \leq i < j \leq 3$, then the formula (2.2.3.1) indicates that $d_3(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}) = 0_{A_4(G)}$, and in all other cases, $d_3(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}})$ forms a combination of NBC-monomials. Hence, for any $1 < k \leq \ell$ and $k \neq t$, $\nexists a_{e_1} a_{e_t} a_{e_k} \in \ker d_3$. But $\nexists a_{e_1} a_{e_t} a_{e_k} \in \text{Im } d_2$, as shown in proposition (2.2.2). Consequently, $\ker d_3 = \text{Im } d_2$ and $\dim(\text{Im } d_3) = \binom{\ell-2}{3} + \binom{\ell-2}{2}$.

2. In case $c(G) = 5$, there are two possibilities:

If $m = 5$, then G is 5-generic. Assume $\mathcal{E} = \{e_1 = [v_1, v_2], e_2 = [v_2, v_3], e_3 = [v_1, v_4], e_4 = [v_3, v_4], e_5 = [v_4, v_5]\}$. So, the broken 5-cycle with no chord written as:

$$a_{e_2} a_{e_3} a_{e_4} a_{e_5} = a_{e_1} a_{e_3} a_{e_4} a_{e_5} - a_{e_1} a_{e_2} a_{e_4} a_{e_5} + a_{e_1} a_{e_2} a_{e_3} a_{e_5} - a_{e_1} a_{e_2} a_{e_3} a_{e_4} \dots (2.2.3.2)$$

Below are four cases for values of t :

If $t = 2$, then:

$$\begin{aligned}
d_3(a_{e_1} a_{e_2} a_{e_3}) &= d_3(a_{e_1} a_{e_2} a_{e_4}) = d_3(a_{e_1} a_{e_2} a_{e_5}) = 0_{A_4(G)}, \\
d_3(a_{e_1} a_{e_3} a_{e_4}) &= d_3(a_{e_2} a_{e_3} a_{e_4}) = -a_{e_1} a_{e_2} a_{e_3} a_{e_4}, \\
d_3(a_{e_1} a_{e_3} a_{e_5}) &= d_3(a_{e_2} a_{e_3} a_{e_5}) = -a_{e_1} a_{e_2} a_{e_3} a_{e_5}, \\
d_3(a_{e_1} a_{e_4} a_{e_5}) &= d_3(a_{e_2} a_{e_4} a_{e_5}) = -a_{e_1} a_{e_2} a_{e_4} a_{e_5}, \\
\text{and } d_3(a_{e_3} a_{e_4} a_{e_5}) &= d_3(a_{e_1} a_{e_3} a_{e_4}) - d_3(a_{e_1} a_{e_3} a_{e_5}) + d_3(a_{e_1} a_{e_4} a_{e_5}).
\end{aligned}$$

If $t = 3$, then:

$$\begin{aligned}
d_3(a_{e_1} a_{e_2} a_{e_3}) &= d_3(a_{e_1} a_{e_3} a_{e_4}) = d_3(a_{e_1} a_{e_3} a_{e_5}) = 0_{A_4(G)}, \\
d_3(a_{e_1} a_{e_2} a_{e_4}) &= -d_3(a_{e_2} a_{e_3} a_{e_4}) = a_{e_1} a_{e_2} a_{e_3} a_{e_4}, \\
d_3(a_{e_1} a_{e_2} a_{e_5}) &= -d_3(a_{e_2} a_{e_3} a_{e_5}) = a_{e_1} a_{e_2} a_{e_3} a_{e_5}, \\
d_3(a_{e_1} a_{e_4} a_{e_5}) &= d_3(a_{e_3} a_{e_4} a_{e_5}) = -a_{e_1} a_{e_3} a_{e_4} a_{e_5}, \\
\text{and } d_3(a_{e_2} a_{e_4} a_{e_5}) &= d_3(a_{e_1} a_{e_2} a_{e_4}) - d_3(a_{e_1} a_{e_2} a_{e_5}) + d_3(a_{e_1} a_{e_4} a_{e_5}).
\end{aligned}$$

If $t = 4$, then:

$$\begin{aligned}
d_3(a_{e_1}a_{e_2}a_{e_4}) &= d_3(a_{e_1}a_{e_3}a_{e_4}) = d_3(a_{e_1}a_{e_4}a_{e_5}) = 0_{A_4(G)}, \\
d_3(a_{e_1}a_{e_2}a_{e_3}) &= d_3(a_{e_2}a_{e_3}a_{e_4}) = -a_{e_1}a_{e_2}a_{e_3}a_{e_4}, \\
d_3(a_{e_1}a_{e_2}a_{e_5}) &= -d_3(a_{e_2}a_{e_4}a_{e_5}) = a_{e_1}a_{e_2}a_{e_4}a_{e_5}, \\
d_3(a_{e_1}a_{e_3}a_{e_5}) &= -d_3(a_{e_3}a_{e_4}a_{e_5}) = a_{e_1}a_{e_3}a_{e_4}a_{e_5}, \\
\text{and } d_3(a_{e_2}a_{e_3}a_{e_5}) &= d_3(a_{e_1}a_{e_2}a_{e_3}) - d_3(a_{e_1}a_{e_2}a_{e_5}) - d_3(a_{e_1}a_{e_3}a_{e_5}).
\end{aligned}$$

If $t = 5$, then:

$$\begin{aligned}
d_3(a_{e_1}a_{e_2}a_{e_5}) &= d_3(a_{e_1}a_{e_3}a_{e_5}) = d_3(a_{e_1}a_{e_4}a_{e_5}) = 0_{A_4(G)}, \\
d_3(a_{e_1}a_{e_2}a_{e_3}) &= d_3(a_{e_2}a_{e_3}a_{e_5}) = -a_{e_1}a_{e_2}a_{e_3}a_{e_5}, \\
d_3(a_{e_1}a_{e_2}a_{e_4}) &= d_3(a_{e_2}a_{e_4}a_{e_5}) = -a_{e_1}a_{e_2}a_{e_4}a_{e_5}, \\
d_3(a_{e_1}a_{e_3}a_{e_4}) &= d_3(a_{e_3}a_{e_4}a_{e_5}) = -a_{e_1}a_{e_3}a_{e_4}a_{e_5}, \\
\text{and } d_3(a_{e_2}a_{e_3}a_{e_4}) &= d_3(a_{e_1}a_{e_2}a_{e_3}) - d_3(a_{e_1}a_{e_2}a_{e_4}) + d_3(a_{e_1}a_{e_3}a_{e_4}).
\end{aligned}$$

Hence, for any $1 < k \leq \ell$ and $k \neq t$, $\mp a_{e_1}a_{e_t}a_{e_k} \in \ker d_3$. But $\mp a_{e_1}a_{e_t}a_{e_k} \in \text{Im } d_2$, as shown in proposition (2.2.2). Consequently, $\ker d_3 = \text{Im } d_2$. It is clear, $\dim(\text{Im } d_3) = \binom{5-2}{3} + \binom{5-2}{2} - 1 = 3$ for this case.

If $m > 5$ and $c_5(G)$ be the number of chordless 5-cycles then for this case,

$$\begin{aligned}
|NBC_{\trianglelefteq G}^3(M_G)| &= |S_{\trianglelefteq G}^3(\Pi^G)| = \binom{\ell}{3}, |BC_{\trianglelefteq G}^4(M_G)| = c_5(G) \text{ and;} \\
|NBC_{\trianglelefteq G}^4(M_G)| &= |S_{\trianglelefteq G}^4(\Pi^G)| - |BC_{\trianglelefteq G}^4(M_G)| = \binom{\ell}{4} - c_5(G).
\end{aligned}$$

Suppose $S = \{e_{q_1}, e_{q_2}, e_{q_3}, e_{q_4}, e_{q_5}\}$ be a chordless 4-cycle such that $1 \leq q_1 < q_2 < q_3 < q_4 < q_5 \leq \ell$. Hence, it's broken cycle is $C = \{e_{q_2}, e_{q_3}, e_{q_4}, e_{q_5}\} \in BC_{\trianglelefteq G}^3(M_G)$. It is known that every monomial of a broken 5-circuit written as a combination of NBC-monomials as:

$$\begin{aligned}
a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}}a_{e_{q_5}} &= a_{e_{q_1}}a_{e_{q_3}}a_{e_{q_4}}a_{e_{q_5}} - a_{e_{q_1}}a_{e_{q_2}}a_{e_{q_4}}a_{e_{q_5}} + a_{e_{q_1}}a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_5}} - \\
&\quad a_{e_{q_1}}a_{e_{q_2}}a_{e_{q_3}}a_{e_{q_4}} \dots \quad (2.2.3.3)
\end{aligned}$$

Therefore, we have the following possible cases:

If S is one of $(c_5 - u_5)$ of chordless 4-cycles that do not simultaneously include e_1 and e_t , then:

- b.1.1.** If $e_1, e_t \notin S$, then formula (2.2.3.1) shows that $d_3(a_{e_1}a_{e_{q_i}}a_{e_{q_j}})$ and $d_3(a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}})$ are written as a combination of NBC-monomials, for $1 \leq i < j < k \leq 5$.
- b.1.2.** For the second case assume either $(e_1 \in S \text{ and } e_t \notin S)$ or $(e_1 \notin S \text{ and } e_t \in S)$. If $(e_1 \in S \text{ and } e_t \notin S)$, then, $q_1 = 1$ and $q_2, q_3, q_4, q_5 \neq t$. Accordingly to formula (2.2.3.1), we get either $d_3(a_{e_1}a_{e_{q_i}}a_{e_{q_j}}) = \mp d_3(a_{e_{q_i}}a_{e_{q_j}}a_{e_t}) = \mp a_{e_1}a_{e_{q_i}}a_{e_{q_j}}a_{e_t}$ is an NBC monomial or $d_2(a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}}) = a_{e_1}a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}} \mp a_{e_{q_i}}a_{e_{q_j}}a_{e_{q_k}}a_{e_t}$, is written as a combination of an NBC monomials, for $1 < i < j < k \leq 5$. Therefore, the number of generators that combine the basis of $\text{Im } d_3$ is 4, determined as: $\binom{5-2}{3} + \binom{5-2}{2} = 4$. Similarly, it's possible

to conclude that four generators will be obtained to be included in $\text{Im } d_3$'s basis, if $(e_1 \notin S \text{ and } e_t \in S)$.

Assume S is one of the u_5 chordless 5-cycles, that including e_1 and e_t , for the purpose of seeing how one of the 2-NBC monomial images cannot be including in $\text{Im } d_2$'s basis, as follows:

If $t = q_2$:

$$d_3(a_{e_{q_3}} a_{e_{q_4}} a_{e_{q_5}}) = d_3(a_{e_1} a_{e_{q_2}} a_{e_{q_4}}) - d_3(a_{e_1} a_{e_{q_3}} a_{e_{q_5}}) + d_3(a_{e_1} a_{e_{q_4}} a_{e_{q_5}}).$$

If $t = q_3$:

$$d_3(a_{e_{q_2}} a_{e_{q_4}} a_{e_{q_5}}) = d_3(a_{e_1} a_{e_{q_2}} a_{e_{q_4}}) - d_3(a_{e_1} a_{e_{q_2}} a_{e_{q_5}}) + d_3(a_{e_1} a_{e_{q_4}} a_{e_{q_5}}).$$

If $t = q_4$:

$$d_3(a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_5}}) = d_3(a_{e_1} a_{e_{q_2}} a_{e_{q_3}}) - d_3(a_{e_1} a_{e_{q_2}} a_{e_{q_5}}) - d_3(a_{e_1} a_{e_{q_3}} a_{e_{q_5}}).$$

If $t = q_5$:

$$d_3(a_{e_{q_2}} a_{e_{q_3}} a_{e_{q_4}}) = d_3(a_{e_1} a_{e_{q_2}} a_{e_{q_3}}) - d_3(a_{e_1} a_{e_{q_2}} a_{e_{q_4}}) + d_3(a_{e_1} a_{e_{q_3}} a_{e_{q_4}}).$$

In this case, the generators joining the basis of $\text{Im } d_3$, are exactly three elements calculated as: $\binom{5-2}{3} + \binom{5-2}{2} - 1 = 3$. For general case, the number of such generators is, $\binom{\ell-2}{3} + \binom{\ell-2}{2} - u_5(G)$.

If $c(G) = 4$, then,

$$\begin{aligned} |\text{NBC}_{\preceq G}^3(M_G)| &= |S_{\preceq G}^3(\Pi^G)| - |\text{BC}_{\preceq G}^3(M_G)| = \binom{\ell}{3} - c_4(G) \text{ and;} \\ |\text{NBC}_{\preceq G}^4(M_G)| &= |S_{\preceq G}^4(\Pi^G)| - (|\text{BC}_{\preceq G}^4(M_G)| + c_4(G)) = \binom{\ell}{4} - (c_5(G) + c_4(G)). \end{aligned}$$

where $c_4(G)$ and $c_5(G)$ be the numbers of chordless 4-cycles and chordless 5-cycles respectively. Here, we need to know wither the G is generic or not:

- a. If G is generic, then $\mathbf{A}_4(G) = 0$. Thus, $\ker d_3 = \mathbf{A}_3(G)$, since $d_3: \mathbf{A}_3(G) \rightarrow 0$ is the zero *homomorphism*.
- b. If G is not 4-generic (i.e. $c_4(G) > 1$), we will be partitioned $[\text{NBC}]_{\preceq G}^{3,0}(M_G)$ by using the type of the chordless 4 cycles. Put $[\text{NBC}]_{\preceq G}^{3,0}(M_G)$ to be the set of all NBC bases that not related to any chordless 4-cycles or related to chordless 4 cycle that contain e_{-1} or $(e_{-t}$ as minimal edge). Put $[\text{NBC}]_{\preceq G}^{3,t}(M_G)$ and $[\text{BC}]_{\preceq G}^{3,t}(M_G)$ to be the set of all NBC bases and broken circuits that related to chordless 4 cycle that contains e_{-t} as not minimal edge via $\preceq_G$, respectively, and let $v_{-4}(G) = |\text{BC}_{\preceq G}^{3,t}(M_G)|$. Put $\text{NBC}_{\preceq G}^{3,1,t}(M_G)$ be the set of all NBC bases that related to chordless 4-cycles that contains each of e_{-1} and e_{-t} . It is clear:

$$\text{NBC}_{\preceq G}^3(M_G) = \text{NBC}_{\preceq G}^{3,0}(M_G) \cup \text{NBC}_{\preceq G}^{3,t}(M_G) \cup \text{NBC}_{\preceq G}^{3,1,t}(M_G).$$

Now, we will study the behavior of d_3 as follows:

- b.1.** If $c_5(G) = 0$, then $BC_{\trianglelefteq G}^4(M_G) = \emptyset$ since every 4 edges either they are in a chordless 4-cycle or they cannot be broken circuit for a chordless 5-cycle. Thus, for this case $NBC_{\trianglelefteq G}^4(M_G) = S_{\trianglelefteq G}^4(\Pi^G) - C_{\trianglelefteq G}^4(M_G)$, where $C_{\trianglelefteq G}^4(M_G)$ is the set of all chordless 4-cycles. According to the formula (2.2.3.1):
- b.1.1.** For $1 < k \leq \ell$ and $k \neq t$, $\{e_1, e_k, e_t\} \in NBC_{\trianglelefteq G}^3(M_G)$ and $d_3(\pm a_{e_1} a_{e_k} a_{e_t}) = 0_{A_4(G)}$. These monomials contained in the basis of Imd_2 , and their number is $(\ell - 2)$.
- b.1.2.** For $1 < k_1 < k_2 \leq \ell$ and $k_1, k_2 \neq t$, if $\{e_1, e_{k_1}, e_{k_2}\} \in NBC_{\trianglelefteq G}^{3,1,t}(M_G)$ then $d_3(a_{e_1} a_{e_{k_1}} a_{e_{k_2}}) = 0_{A_4(G)}$. There are $u_4(G)$ such monomials, and Imd_2 does not include them.
- b.1.3.** If $\{e_{(k_1)}, e_{(k_2)}, e_{(k_3)}\} \in \llbracket NBC \rrbracket_{\trianglelefteq G}^{3,0}(M_G)$, such that $1 < k_1 < k_2 < k_3 \leq \ell$ and $k_1, k_2, k_3 \neq t$, as a direct result to formula (2.2.3.1), each of $d_3(a_{(e_1)} a_{(e_{(k_1)})} a_{(e_{(k_2)})})$, $d_3(a_{(e_1)} a_{(e_{(k_1)})} a_{(e_{(k_3)})})$, $d_3(a_{(e_1)} a_{(e_{(k_2)})} a_{(e_{(k_3)})})$ are written as a combination of NBC monomials.
- b.1.4.** If $\{e_{k_1}, e_{k_2}, e_{k_3}\} \in NBC_{\trianglelefteq G}^{3,t}(M_G)$, such that $1 < k_1 < k_2 < k_3 \leq \ell$ and $k_1, k_2, k_3 \neq t$, hence $d_3(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}) = -a_{e_1} a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}$ is an NBC monomials and,

$$d_3(a_{e_1} a_{e_{k_2}} a_{e_{k_3}}) = d_3(a_{e_1} a_{e_{k_1}} a_{e_{k_3}}) - d_3(a_{e_1} a_{e_{k_1}} a_{e_{k_2}}) + d_3(a_{e_{k_1}} a_{e_{k_2}} a_{e_{k_3}}).$$

The number of such monomials is $|NBC_{\trianglelefteq G}^{3,t}(M_G)| = v_4$.

If $B = \{e_1, e_{k_1}, e_{k_2}\} \in NBC_{\trianglelefteq G}^{3,1,t}(M_G)$, $1 < k_1 < k_2 \leq \ell$ and $k_1, k_2 \neq t$, then for all $1 < k \leq \ell$ with $k \neq k_1, k_2, t$ and $\{e_{k_1}, e_{k_2}, e_k\}$ not a broken 3-circuits;

$$d_3(a_{e_{k_1}} a_{e_{k_2}} a_{e_k}) = \pm d_3(a_{e_1} a_{e_{k_2}} a_{e_k}) \mp d_3(a_{e_1} a_{e_{k_1}} a_{e_k}).$$

The number of such monomials is $(u_4(\ell - 4) - b_4)$, where b_4 is the number of broken circuits that contains $B - \{e_1\}$, and $B \in NBC_{\trianglelefteq G}^{3,1,t}(M_G)$.

For this case, $\dim(\ker d_3) = \dim(Imd_2) + u_4(G)$ and:

$$\dim(Imd_3) = \left(\binom{\ell - 2}{3} - (u_4(\ell - 4) - b_4) - (c_4 - u_4) \right) + \left(\binom{\ell - 2}{2} - (u_4 + v_4) \right).$$

The discussion of the case that $c_5(G) \neq 0$ is a mixture among all the cases that we analyzed in item 2 ($c(G) = 5$) and item b.1 ($c(G) = 4$ and $c_5(G) = 0$) above. Thus, $\dim(\ker d_3) = \dim(Imd_2) + u_4$ and:

$$\dim(Imd_3) = \left(\binom{\ell - 2}{3} - (u_4(\ell - 4) - b_4) - (c_4 - u_4) \right) + \left(\binom{\ell - 2}{2} - (u_4 + v_4) \right) - u_5$$

Theorem

If $a = a_{e_1} - a_{e_t}$, for $2 \leq t \leq \ell$, then:

1. The Orlik-Solomon algebra $\mathbf{A}_*(G)$ has vanished first cohomological group, i.e. $H^1(\mathbf{A}_*(G); a) = 0$.
2. The structure of $H^2(\mathbf{A}_*(G); a)$ depending on the value of $c(G)$, as follows:
 - i. If G is tree or $c(G) \geq 5$, then $\mathbf{A}_*(G)$ has vanished $H^2(\mathbf{A}_*(G); a)$.
 - ii. If $c(G) = 4$, then the Orlik-Solomon algebra $\mathbf{A}_*(G)$ has non vanished second cohomological group with $\dim(H^2(\mathbf{A}_*(G); a)) = u_4(G)$, where $u_4(G)$ be the number of chordless 4-cycles that includes e_1 and e_t .

Proof

This is a direct result of proposition (2.2.2) and theorem (2.2.3).

Illustrations

We will illustrate our results as follows:

Trees

All the trees in figure (1) are hypersolvable with 15 vertices, 14 edges and each one of them has hypersolvable partition with exponent vector $(1, 1, \dots, 1)$. According to theorem (2.2.4), they have vanished first and second cohomology of the Orlik-Solomon algebra:

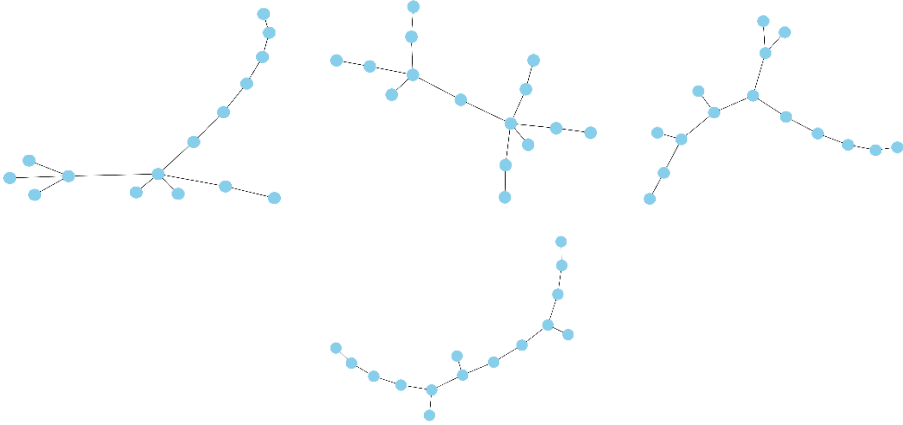


Figure 1: Trees with 15 vertices and 14 edges.

Hypercubes

Figure 2 illustrates hypercubes of dimensions 2, 3, and 4, each of which has a hypersolvable partition with an exponent vector of $(1, 1, \dots, 1)$, as demonstrated in construction (2.1.1). The first cohomology of the Orlik-Solomon algebra has vanished as a result of the application of the theorem (2.2.4). The vanishing of the second cohomology of the Orlik-Solomon algebra is contingent upon the selection of $2 \leq t \leq \ell$, where $\ell = 4, 6, 24$ related to the dimension of the hypercube 2, 3, 4, as follows:

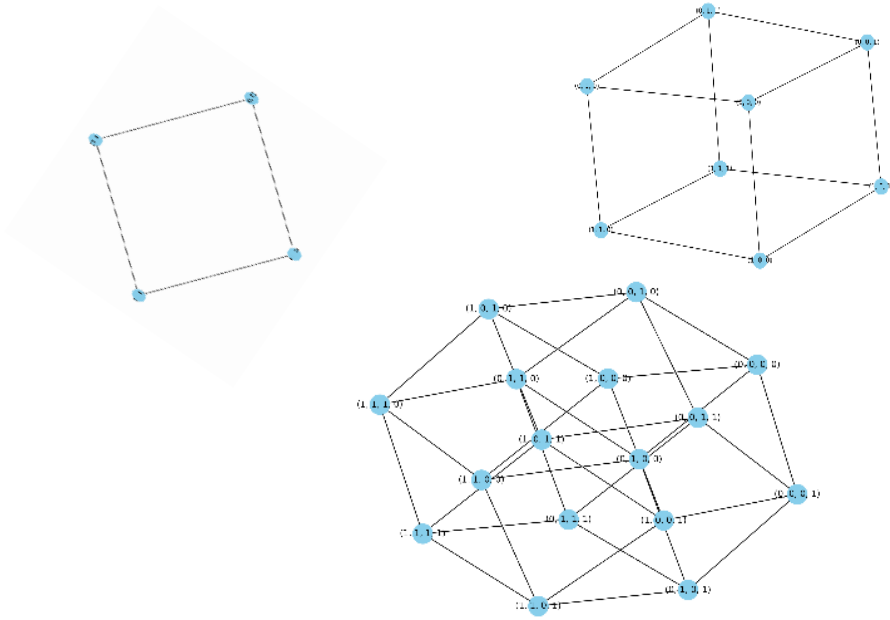


Figure 2: Hypercubes of dimension 2, 3 and 4 respectively.

1. For the hypercube G of dimensions 2, we have four edges and just one chordless 4-cycle, $\{e_1, e_2, e_3, e_4\}$ via the hypersolvable order. Thus $\dim(H^2(\mathbf{A}_*(G); a_{e_1} - a_{e_t})) = 1$, for $2 \leq t \leq 4$.
2. If dimension of the hypercube G is 3, then it has twelve edges and six chordless 4-cycles $\{e_1, e_2, e_4, e_5\}$, $\{e_1, e_3, e_6, e_7\}$, $\{e_2, e_3, e_8, e_9\}$, $\{e_4, e_6, e_{10}, e_{11}\}$, $\{e_5, e_8, e_{10}, e_{12}\}$ and $\{e_7, e_9, e_{11}, e_{12}\}$ via the hypersolvable order. There is just one chordless 4-cycle that can contain e_1 and e_t , for $2 \leq t \leq 7$ and there is no chordless 4-cycle contains e_1 and e_t , for $8 \leq t \leq 12$. Therefore:

$$\dim(H^2(\mathbf{A}_*(G); a_{e_1} - a_{e_t})) = \begin{cases} 1 & : 2 \leq t \leq 7 \\ 0 & : 8 \leq t \leq 12 \end{cases}$$

3. The hypercube G of dimension 4, has thirty-two edges and twenty four chordless 4-cycles $\{e_1, e_2, e_5, e_6\}$, $\{e_1, e_3, e_7, e_8\}$, $\{e_1, e_4, e_9, e_{10}\}$, $\{e_2, e_3, e_{11}, e_{12}\}$, $\{e_2, e_4, e_{13}, e_{14}\}$, $\{e_3, e_4, e_{15}, e_{16}\}$, $\{e_5, e_7, e_{17}, e_{18}\}$, $\{e_5, e_9, e_{20}, e_{21}\}$, $\{e_6, e_{11}, e_{17}, e_{19}\}$, $\{e_6, e_{13}, e_{20}, e_{22}\}$, $\{e_7, e_9, e_{23}, e_{24}\}$, $\{e_8, e_{12}, e_{18}, e_{19}\}$, $\{e_8, e_{15}, e_{23}, e_{25}\}$, $\{e_{10}, e_{11}, e_{21}, e_{22}\}$, $\{e_{10}, e_{16}, e_{24}, e_{25}\}$, $\{e_{11}, e_{13}, e_{26}, e_{27}\}$, $\{e_{12}, e_{15}, e_{26}, e_{28}\}$, $\{e_{14}, e_{16}, e_{27}, e_{28}\}$, $\{e_{17}, e_{20}, e_{29}, e_{30}\}$, $\{e_{18}, e_{23}, e_{29}, e_{31}\}$, $\{e_{19}, e_{26}, e_{29}, e_{32}\}$, $\{e_{21}, e_{24}, e_{30}, e_{31}\}$, $\{e_{22}, e_{27}, e_{30}, e_{32}\}$ and $\{e_{25}, e_{28}, e_{31}, e_{32}\}$ via the hypersolvable order. Clearly, there is just one chordless 4-cycle that can contain e_1 and e_t , for $2 \leq t \leq 10$ and there is no chordless 4-cycle contains e_1 and e_t , for $11 \leq t \leq 32$. Therefore:

$$\dim(H^2(\mathbf{A}_*(G); a_{e_1} - a_{e_t})) = \begin{cases} 1 & : 2 \leq t \leq 10 \\ 0 & : 11 \leq t \leq 32 \end{cases}$$

Conclusion

This study is intended to investigate the first non-vanishing cohomology $H^*(\mathbf{A}_*(G); a)$, of the Orlik-Solomon algebra $\mathbf{A}_*(G)$ for a free-tringles graph G such that $a = a_{e_1} - a_{e_t}$, $2 \leq t \leq \ell$, specifically that has chordless 4-cycles and conclusion was driven based on the results:

1. The graph G not containing triangles was the reason for the vanishing of its first cohomology $H^1(\mathbf{A}_*(G); a)$ of the Orlik-Solomon algebra.
2. If a graph G with no chordless 4-cycles, then $H^2(\mathbf{A}_*(G); a)$ vanished.
3. If G has no chordless 4-cycles that includes e_1 and e_t , then $H^2(\mathbf{A}_*(G); a)$ vanished.
4. If the number of chordless 4-cycles that includes e_1 and e_t , $u_4(G) > 0$, then $\dim(H^2(\mathbf{A}_*(G); a)) = u_4(G)$.

Our work reconstructs the Orlik-Solomon algebra by combining arrangements and graphs using matroids. We may investigate the implications of these results for a variety of networks and their configurations in the future. This might generate new ideas and applications in related mathematical fields.

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