

ON *MP-CLEAN ELEMENT & MP-CLEAN RING*

Mohammed Dawood Salman^{1*}, Wissam Abbas Hussein², Ali Kadhim Yaqoob³

Abstract: Impose that $\mathcal{C}$ is unitary ring in this work paper we define another generalization of clean ring name mp-clean ring and show the connection between this concept and another clean rings (p-clean, m-clean, mj-clean). Where $\mathcal{C}$ is mp-clean ring if all elements in $\mathcal{C}$ are mp-clean elements. some examples and properties of mp-clean ring gives, and we define another element say mp-clean element.

Introduction

The concept clean ring given by Chen and Cui [2] and Nicholson [7] where $\mathcal{C}$ is clean ring if there exist unit element in $\mathcal{U}(\mathcal{C})$ * $\mathcal{U}(\mathcal{C})$ refer to the set of all unit element in $\mathcal{C}^*$ and idempotent element in $\mathcal{E}(\mathcal{C})$ where $\mathcal{E}(\mathcal{C}) = \{\forall e \in \mathcal{C}, e^2 = e\}$ then $\mathcal{C}$ is the sum of (unit & idempotent). $\mathcal{U}(\mathcal{C}) = \{\forall v \in \mathcal{C}, \exists k \in \mathcal{C} \text{ s.t. } v.k = k.v = 1\}$. Also Mohammed, et al. [6] introduced another concept is *p-clean ring* where $\mathcal{C}$ is *p-clean ring* if all elements in $\mathcal{C}$ c is p-clean i.e. $c = e + p$ for some $e \in \mathcal{E}(\mathcal{C})$ and $p \in \mathcal{P}(\mathcal{C})$ * $\mathcal{P}(\mathcal{C})$ is the set of all pure element $\mathcal{P}(\mathcal{C}) = \{\forall p \in \mathcal{C} \exists q \in \mathcal{C} \text{ s.t. } p = p.q\}$. Ashrafi and Nasibi [1] gives another generalization name r-clean element and r-clean ring' where for all c in $\mathcal{C}$ there exist $e \in \mathcal{E}(\mathcal{C})$ and r in $\text{reg}(\mathcal{C})$ such that $c = e + r$ then c is r-clean element so that $\mathcal{C}$ is called r-clean ring if all elements in $\mathcal{C}$ is r-clean element. * $\text{reg}(\mathcal{C})$ is the set of all regular element*. also Purkait, et al. [8], and Vijayanand and Selvaraj [10], define *m-clean element and m-clean ring* where c is called *m-clean* element if $c = e + v$ for some v in $\mathcal{U}(\mathcal{C})$ and $e^m = e$, s.t $m \geq 2$ is positive integer number so that $\mathcal{C}$ is called m-clean ring if all elements in $\mathcal{C}$ is m-clean element. Esfandiar, et al. [5] and Ulucak and Kör [9] gives *mj-clean element and*mj-clean* ring* where c in $\mathcal{C}$ is *mj-clean element* if $c = e + r$ for some r in $\text{rad}(\mathcal{C})$ and $e^m = e$, s.t $m \geq 2$ is positive integer number Also D'Anna, et al. [4] give another generalization name amalgamated ring $\mathcal{C} \ltimes^{\phi} \mathcal{A} = \{(r, \phi(r) + a, r \in \mathcal{C}, a \in \mathcal{A}\}$ such that $\phi: \mathcal{C} \rightarrow \mathcal{M}$ homeomorphisem and $\mathcal{A}$ be an ideal of $\mathcal{M}$. Now we introduce *mp-clean element and mp-clean ring*.

mp-clean ring

Impose c in $\mathcal{C}$ and $i^m = i$, s.t $m \geq 2$ is positive integer number, then c is mp-clean element If $c = i + p$ for some $p \in \mathcal{P}(\mathcal{C})$ and i is *m-potent element* in $\mathcal{C}$.

¹ Institute of Applied Arts, Middle Technical University, Baghdad, Iraq.
 Email: mohammed_dawood@mtu.edu.iq

² General Directorate of Education of Salahuddin, Salahuddin, Iraq.
 Email: wissam.abbas1987@gmail.com

³ Department of Mathematics, College of Science, University of Anbar; Ramadi, Iraq.
 Email: ali.kazem@uoanbar.edu.iq

*Correspondence: mohammed_dawood@mtu.edu.iq

Definition

$\mathcal{C}$ name mp-clean ring if all elements in $\mathcal{C}$ are mp-clean element.

Examples

1. The ring $(\mathbb{Z}_8, +_8, \cdot_8)$ is mp-clean ring. since $\mathbb{Z}_8$ has *m-potent elements $=\{\bar{0}, \bar{1}\}$ and pure elements $=\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$ satisfy the condition each element in $\mathbb{Z}_8$ is *mp- clean element*.
2. The ring $(\mathbb{Z}_6, +_6, \cdot_6)$ is mp-clean ring. since $\mathbb{Z}_6$ has *m-potent elements $=\{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}$ and pure elements $=\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ satisfy the condition each element in $\mathbb{Z}_6$ is *mp-clean element*.
3. $(\mathbb{Z}, +, \cdot)$ is *mp-clean*.
4. If all element in $\mathcal{C}$ are m-potent element, then $\mathcal{C}$ is *mp- clean ring*.
5. The matrix $M_3(\mathbb{Z}_3)$ is mp-clean, let $\kappa = \begin{bmatrix} a & b & c \\ d & e & f \\ j & i & h \end{bmatrix}$ belong to the matrix M_3 then there exist identity matrix $i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ such that $\begin{bmatrix} a & b & c \\ d & e & f \\ j & i & h \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ j & i & h \end{bmatrix}$ so that every element in M_3 is pure element hence $p = \begin{bmatrix} a-1 & b & c \\ d & e-1 & f \\ j & i & h-1 \end{bmatrix}$ in M_3 is pure element.

Now we must show that $e^m = e$ in M_3 , since $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = e$ in M_3 so that $e^m = e$ for all positive

integer number $m \geq 2$ then we can written

$\kappa = p + e \begin{bmatrix} a & b & c \\ d & e & f \\ j & i & h \end{bmatrix} = \begin{bmatrix} a-1 & b & c \\ d & e-1 & f \\ j & i & h-1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that κ is *mp-clean* then $M_3(\mathbb{Z}_3)$ is *mp- clean*

Proposition 2.3

impose $\mathcal{C}$ is *clean ring* then it is *mp-clean ring* for all $m \geq 2$ is positive integer number but the converse is not true.

Proof: Impose that $\mathcal{C}$ is *clean ring* so if κ in $\mathcal{C}$ then $\kappa = v + e$ s.t there exist $v \in \hat{U}(\mathcal{C})$ & $e \in \hat{E}(\mathcal{C})$ to show that $\mathcal{C}$ is *mp-clean ring* we must prove all elements in $\mathcal{C}$ is mp-clean i.e. $\kappa = e + p$ for some $p \in \mathcal{P}(\mathcal{C})$ e is m-potent element in $\mathcal{C}$ so we should prove that e is m-potent and $u=p$. Now since $e \in \hat{E}(\mathcal{C})$ so that $e^2=e$ then clearly e is m-potent $\forall m \geq 2$ is positive integer number, now since $v \in \hat{U}(\mathcal{C})$ so there exist w in $\mathcal{C}$ such that $v \cdot w = w \cdot v = 1$. since $1 \in \mathcal{C}$ so that $v = v \cdot 1$ therefore $v \in \mathcal{P}(\mathcal{C})$ so $v=p$ hence $\mathcal{C}$ is *mp- clean ring*.

For conversely it is not success $(\mathbb{Z}, +, \cdot)$ is *mp-clean* but not clean.

Proposition 2.4

Spouse that $\mathcal{C}$ is *P-clean* then $\mathcal{C}$ is *mp-clean* for all $m \geq 2$ is positive integer number

Proof: Impose that $\mathcal{C}$ is *P- clean ring* so all element is written as $\kappa = p + e$ for some $p \in \mathcal{P}(\mathcal{C})$ & $e \in \hat{E}(\mathcal{C})$ to show that $\mathcal{C}$ is *mp-clean ring* we must prove all elements in $\mathcal{C}$ is

mp-clean i.e. $\kappa = i + p$ for some $p \in P(C)$ and i is *m-potent element* in C so we should prove that e is *m -potent* and. Now since $e \in \hat{E}(C)$ so that $e^2 = e \forall m \geq 2$ is positive integer number then clearly e is *m-potent* hence C is *mp - clean ring*.

Proposition 2.5

Assume that C is *m -clean* then C is *mp-clean* for all $m \geq 2$ is positive integer number

proof: Impose that C is *m- clean ring* so if κ in C then $\kappa = v + e$ for some $v \in \hat{U}(C)$ & $e^2 = e \forall m \geq 2$ is positive integer number, e is m-potent for all $m \geq 2$ is positive integer number to show that C is *mp-clean ring* we must prove all elements in C is mp-clean i.e. $\kappa = e + p$ for some $p \in P(C)$ e is m-potent element in C so we should prove that $v = p$. Now since $v \in \hat{U}(C)$ so there exist w in C such that $v.w = w.v = 1$. since $1 \in C$ so that $v = v.1$ therefore $v \in P(C)$ so $v = p$ hence C is *mp - clean ring*.

Corollary

If C is *mj-clean* then C is *mp-clean* for all $m \geq 2$ is positive integer number

Proof: Impose that C is *mj- clean ring* if κ in C then $\kappa = j + e$ s.t there exist $j \in \text{rad}(C)$ and $e^m = e \forall m \geq 2$ is positive integer number, e is *m -potent* since every *mj -clean ring* is *m -clean ring* therefore prove clearly by proposition.2.5

Proposition 2.7

If $\phi: C \rightarrow C'$ be epimorphism mapping and C is *mp-clean ring* for all $m \geq 2$ is positive integer number then C' is *mp-clean ring*.

Proof: Impose that C is *mj - clean ring* and let $\kappa' \in C'$ so $\exists \kappa \in C$, s. t, $\phi(\kappa) = \kappa'$. since C is *mj- clean ring* so there exist p in $P(C)$ and $e^m = e \forall m \geq 2$ is positive integer number, e is m-potent. so $\kappa' = \phi(e + p) = \phi(e) + \phi(p)$, now $\phi(e^m) = [\phi(e)]^m$ because $e^m = e \forall m \geq 2$ is positive integer number, e is m-potent so $\phi(e^m)$ is m -potent.

Now we prove $\phi(p)$ in $P(C)$ since $p \in P(C)$ so there exist b in C such that $p = pb$ hence $\phi(p) = \phi(pb) = \phi(p)\phi(b)$ therefore $\phi(b)$ is pure element so that C' is *mp-clean ring*.

Proposition 2.8

Assume that $\phi: C \rightarrow M$ homeomorphism mapping and A an ideal of M and then If amalgamated ring $C \rtimes^\phi A$ is *uniquely mp-clean ring* then C and $\phi(C) + A$ is *mp-clean ring (uniquely)* for all $m \geq 2$ is positive integer number.

Proof: Impose that amalgamated ring $(BI) C \rtimes^\phi A$ is *mp-clean ring* and let $p + e = b + F$ such that p, b in $P(C)$ and e, F are m-potent in C then $(p, \phi(p)) + (e, (\phi(e))) = (b, \phi(b)) + (F, (\phi(F)))$ but $(p, \phi(p))$ and $(b, \phi(b))$ are pure in $C \rtimes^\phi A$ and $(e, (\phi(e)))$, $(F, (\phi(F)))$ are m-potent in $C \rtimes^\phi A$ since $(p, \phi(p)) = (b, \phi(b))$ and $(e, (\phi(e))) = (F, (\phi(F)))$ so $p = b$ is pure in C and $e = F$ is m-potent in C thus C is *mp-clean ring* for all $m \geq 2$ is positive integer number.

Now since $C \rtimes^\phi A$ is *mp-clean ring* and its a homeomorphism whit C and $\phi(C) + A$ so that C and $\phi(C) + A$ are *mp-clean ring*.

Corollary

If amalgamated ring $C \rtimes^\phi J$ is *m-clean* then $\phi(C) + J$ is *mp-clean* for all $m \geq 2$ is

positive integer number.

Proof: Since $\mathcal{C} \rtimes^{\mathcal{F}} \mathcal{J}$ is *m-clean ring* then $\mathcal{F}(\mathcal{C}) + \mathcal{J}$ is *m-clean ring* by [5, proposition 3.2] so $\mathcal{F}(\mathcal{C}) + \mathcal{J}$ is *mp-clean* it is clearly by proposition.2.5.

Proposition 2.10

Impose that $\mathcal{C}$ is *mp-clean* then $\mathcal{C}/\mathcal{A}$ is *mp-clean* where $\mathcal{A}$ any ideal in $\mathcal{C}$.

Proof: We must prove that $\kappa + \mathcal{A} = (\mathfrak{p} + \mathcal{A}) + (\mathfrak{e} + \mathcal{A})$ is *mp-clean element* for some pure element $(\mathfrak{p} + \mathcal{A})$ in $\mathcal{C}/\mathcal{A}$, $(\mathfrak{e} + \mathcal{A})$ is m-potent in $\mathcal{C}/\mathcal{A}$.

Now since $\mathcal{C}$ is *mp-clean ring* then $\kappa = \mathfrak{p} + \mathfrak{e}$ in $\mathcal{C}$ is *mp-clean* for some $\mathfrak{p} \in \mathcal{P}(\mathcal{C})$ and $\mathfrak{e} \mathfrak{m} = \mathfrak{e} \forall \mathfrak{m} \geq 2$ is positive integer number, $\mathfrak{e}$ is m-potent. Since $\mathfrak{p} \in \mathcal{P}(\mathcal{C})$ there exist $\mathfrak{b}$ in $\mathcal{C}$ s.t $\mathfrak{p} = \mathfrak{p} \mathfrak{b}$ hence $\mathfrak{p} + \mathcal{A} = \mathfrak{p} \mathfrak{b} + \mathcal{A} = (\mathfrak{p} + \mathcal{A})(\mathfrak{b} + \mathcal{A})$ so $\mathfrak{p} + \mathcal{A}$ is pure in $\mathcal{C}/\mathcal{A}$. now since $\mathfrak{e}^{\mathfrak{m}} = \mathfrak{e}$ then we must show $(\mathfrak{e} + \mathcal{A})^{\mathfrak{m}} = \mathfrak{e} + \mathcal{A}$. so by hypothesis if $\mathfrak{m} = 2$ then $(\mathfrak{e} + \mathcal{A})(\mathfrak{e} + \mathcal{A}) = \mathfrak{e} + \mathcal{A}$. $\mathfrak{e} + \mathcal{A} = \mathfrak{e}^2 + \mathcal{A} = \mathfrak{e} + \mathcal{A}$ because $\mathfrak{e}^2 = \mathfrak{e}$, if $\mathfrak{m} = 3$ then $(\mathfrak{e} + \mathcal{A})(\mathfrak{e} + \mathcal{A})(\mathfrak{e} + \mathcal{A}) = (\mathfrak{e}^2 + \mathcal{A})(\mathfrak{e} + \mathcal{A}) = \mathfrak{e}^3 + \mathcal{A}$ but $\mathfrak{e}^3 = \mathfrak{e}$ then $(\mathfrak{e} + \mathcal{A})^3 = \mathfrak{e} + \mathcal{A}$ therefore by the same way for all $\mathfrak{m} \geq 2$ is positive integer number then $(\mathfrak{e} + \mathcal{A})^{\mathfrak{m}} = \mathfrak{e} + \mathcal{A}$ so $\mathfrak{e} + \mathcal{A}$ is m-potent in $\mathcal{C}/\mathcal{A}$ hence $\mathcal{C}/\mathcal{A}$ is mp.

Proposition 2.11

Let $\mathcal{F} : \mathcal{C} \rightarrow \mathbf{M}$ be a homeomorphism mapping and $\mathcal{A}$ an ideal of $\mathbf{M}$ and let $I_0 = \mathcal{F}^{-1}(\mathcal{A})$ and $\mathcal{C}/I_0$ are unique *mp-clean ring*. then $\mathcal{C} \rtimes^{\mathcal{F}} \mathcal{A}$ is *mp-clean ring* $\Leftrightarrow \mathcal{C}$ and $\mathcal{F}(\mathcal{C}) + \mathcal{A}$ are *mp-clean ring*.

Proof: Impose that $\mathcal{C} \rtimes^{\mathcal{F}} \mathcal{A}$ is *mp-clean ring* so prove clearly by proposition.2.8 then $\mathcal{C}$ and $\mathcal{F}(\mathcal{C}) + \mathcal{A}$ are *mp-clean ring*.

Conversely impose that $\mathcal{C}$ and $\mathcal{F}(\mathcal{C}) + \mathcal{A}$ are *mp-clean ring* to prove that $\mathcal{C} \rtimes^{\mathcal{F}} \mathcal{A}$ is *mp-clean ring* we must show that for every $(\kappa, \mathcal{F}(\kappa) + \mathcal{a})$ there exist $(\mathfrak{p}, \mathcal{F}(\mathfrak{p}) + \mathfrak{a}_1 + \mathfrak{a}'_1)$ is pure and $(\mathfrak{e}, \mathcal{F}(\mathfrak{e}) + \mathfrak{a}_2 + \mathfrak{a}'_2)$ is m-potent, then $(\kappa, \mathcal{F}(\kappa) + \mathcal{a}) = (\mathfrak{p}, \mathcal{F}(\mathfrak{p}) + \mathfrak{a}_1 + \mathfrak{a}'_1) + (\mathfrak{e}, \mathcal{F}(\mathfrak{e}) + \mathfrak{a}_2 + \mathfrak{a}'_2)$ now since $\mathcal{C}$ is mp-clean then $\kappa = \mathfrak{p} + \mathfrak{e}$ for some $\mathfrak{p}$ in $\mathcal{P}(\mathcal{C})$ and $\mathfrak{e}$ is m-potent in $\mathcal{C}$, since $\mathcal{F}(\mathcal{C}) + \mathcal{A}$ is *mp-clean ring* then $\mathcal{F}(\kappa) + \mathcal{a} = \mathcal{F}(\mathfrak{b}) + \mathfrak{a}_1 + \mathcal{F}(\mathfrak{f}) + \mathfrak{a}_2$ for some $\mathcal{F}(\mathfrak{f}) + \mathfrak{a}_2$ is m-potent in $(\mathcal{F}(\mathcal{C}) + \mathcal{A})$ and $\mathcal{F}(\mathfrak{b}) + \mathfrak{a}_1$ is pure in $\mathcal{P}(\mathcal{F}(\mathcal{C}) + \mathcal{A})$ and since $\mathcal{F}(\mathfrak{b}) = \mathcal{F}(\mathfrak{b}) + \mathfrak{a}_1$ and $\mathcal{F}(\mathfrak{f}) = \mathcal{F}(\mathfrak{f}) + \mathfrak{a}_2$ are elements in $\mathcal{P}(\mathcal{F}(\mathcal{C}) + \mathcal{A}/\mathcal{A})$ and m-potent in $(\mathcal{F}(\mathcal{C}) + \mathcal{A}/\mathcal{A})$ so that $\mathcal{F}(\kappa) = \mathcal{F}(\mathfrak{p}) + \mathcal{F}(\mathfrak{e})$ by proposition.5.1 in D'Anna and Fontana [3], hence $\mathcal{C}/I_0 \cong \mathcal{F}(\mathcal{C}) + \mathcal{A}/\mathcal{A}$ so that $\mathcal{C}/I_0 \cong \mathcal{F}(\mathcal{C}) + \mathcal{A}/\mathcal{A}$ it is *uniquely mp-clean ring* therefore $\mathcal{F}(\mathfrak{p}) = \mathcal{F}(\mathfrak{b})$ and $\mathcal{F}(\mathfrak{e}) = \mathcal{F}(\mathfrak{f})$ so that $(\kappa, \mathcal{F}(\kappa) + \mathcal{a}) = (\mathfrak{p}, \mathcal{F}(\mathfrak{p}) + \mathfrak{a}_1 + \mathfrak{a}'_1) + (\mathfrak{e}, \mathcal{F}(\mathfrak{e}) + \mathfrak{a}_2 + \mathfrak{a}'_2)$ since $(\mathfrak{e}, \mathcal{F}(\mathfrak{e}) + \mathfrak{a}_2 + \mathfrak{a}'_2) = \mathcal{F}(\mathfrak{e}) + \mathfrak{a}_2$ is m-potent in $\mathcal{C} \rtimes^{\mathcal{F}} \mathcal{A}$ now just prove $(\mathfrak{p}, \mathcal{F}(\mathfrak{p}) + \mathfrak{a}_1 + \mathfrak{a}'_1)$ in $\mathcal{P}(\mathcal{C} \rtimes^{\mathcal{F}} \mathcal{A})$ since $(\mathcal{F}(\mathfrak{p}) + \mathfrak{a}_1 + \mathfrak{a}'_1)$ is $\mathcal{P}(\mathcal{F}(\mathcal{C}) + \mathcal{A})$ so there exist $\mathcal{F}(\mathfrak{k}) + \mathfrak{a}_0$ such that $(\mathcal{F}(\mathfrak{p}) + \mathfrak{a}_1 + \mathfrak{a}'_1) = (\mathcal{F}(\mathfrak{p}) + \mathfrak{a}_1 + \mathfrak{a}'_1)(\mathcal{F}(\mathfrak{k}) + \mathfrak{a}_0)$ i.e. $\mathcal{F}(\mathfrak{p}) = \mathcal{F}(\mathfrak{p})\mathcal{F}(\mathfrak{k}) = \mathcal{F}(\mathfrak{p})\mathcal{F}(\mathfrak{k})$ so that $(\mathfrak{p}, \mathcal{F}(\mathfrak{p}) + \mathfrak{a}_1 + \mathfrak{a}'_1) = (\mathfrak{p}, \mathcal{F}(\mathfrak{p}) + \mathfrak{a}_1 + \mathfrak{a}'_1)(\mathfrak{k}, \mathcal{F}(\mathfrak{k}) + \mathfrak{a}_0)$ so $(\mathfrak{p}, \mathcal{F}(\mathfrak{p}) + \mathfrak{a}_1 + \mathfrak{a}'_1)$ in $\mathcal{P}(\mathcal{C} \rtimes^{\mathcal{F}} \mathcal{A})$ therefore $\mathcal{C} \rtimes^{\mathcal{F}} \mathcal{A}$ is *mp-clean ring*.

Lemma

Let $\mathcal{F} : \mathcal{C} \rightarrow \mathbf{M}$ be a homeomorphism mapping and $\mathcal{A}$ an ideal of $\mathbf{M}$ and $\mathcal{F}(\mathcal{C}) + \mathcal{A}/\mathcal{A}$ is *uniquely mp-clean ring*. Then $\mathcal{C} \rtimes^{\mathcal{F}} \mathcal{A}$ is *mp-clean ring* $\Leftrightarrow \mathcal{C}$ and is *mp-clean ring*.

Proof: Impose that $\mathcal{C} \rtimes^{\mathcal{F}} \mathcal{A}$ is *mp-clean ring* so prove clearly by proposition.2.8 then $\mathcal{C}$ is *mp-clean ring*.

Conversely: impose that $\mathcal{C}$ is *mp-clean ring* to prove that $\mathcal{C} \ltimes^{\mathcal{A}} \mathcal{A}$ is *mp-clean ring* since κ in $\mathcal{C}$ and $\mathcal{C}$ is *mp-clean ring* then there exist p in $\mathcal{P}(\mathcal{C})$ and e is m-potent element such that $\kappa = p + e$, let $(\kappa, \mathcal{A}(\kappa) + a)$ belong to $\mathcal{C} \ltimes^{\mathcal{A}} \mathcal{A}$ so by proposition.2.12 there exist $(p, \mathcal{A}(p) + a)$ is pure in $\mathcal{C} \ltimes^{\mathcal{A}} \mathcal{A}$ and $(e, \mathcal{A}(e))$ is m-potent in $\mathcal{C} \ltimes^{\mathcal{A}} \mathcal{A}$, such that $(\kappa, \mathcal{A}(\kappa) + a) = (p, \mathcal{A}(p) + a) + (e, \mathcal{A}(e))$ so that $\mathcal{C} \ltimes^{\mathcal{A}} \mathcal{A}$ is *mp-clean ring*.

Proposition 2.13

If $\mathcal{C}$ is *r-clean* then $\mathcal{C}$ is *mp-clean ring* for all $m \geq 2$ is positive integer number.

Proof: Since $\mathcal{C}$ is *r-clean ring* then for all c in $\mathcal{C}$ there exist e in $\hat{E}(\mathcal{C})$ and r in $\text{reg}(\mathcal{C})$ such that $\kappa = e + r$ we must show that $r = p$ since r in $\text{reg}(\mathcal{C})$ so there exist k in $\mathcal{C}$ such that $r = rkr$ take $F = kr$ so that $r = rF$ then r in $\mathcal{P}(\mathcal{C})$ so $r = p$. hence $\mathcal{C}$ is *mp-clean ring* for all $m \geq 2$ is positive integer number.

Remark

The conversely of proposition2.10 is not true for example: $(\mathbb{Z}, +, \cdot)$ is not r-clean but its mp-clean.

Proposition 2.15

If κ is mp-clean in $\mathcal{C}$ then $-\kappa$ is *mp-clean* for each odd positive integer number $m \geq 3$

Proof: Impose that κ is mp in $\mathcal{C}$ then $\kappa = e + p$ for some $p \in \mathcal{P}(\mathcal{C})$ e is m-potent element in $\mathcal{C}$ now... $-\kappa = -(e) + (-p)$ we must prove $-p \in \mathcal{P}(\mathcal{C})$ and $(-e)^m = -e$ is m-potent element in $\mathcal{C}$ for all odd positive integer number $m \geq 3$, since $p \in \mathcal{P}(\mathcal{C})$ so there exist b in $\mathcal{C}$ s.t $p = p b$ so $-p = -p b = (-p)b$ so $-p \in \mathcal{P}(\mathcal{C})$ now since $e^m = e$ then if $m=3$ then $(-e)^3 = -e \cdot -e \cdot -e = -e^3 = -e$ also if we take $m=5$ then $(-e)^5 = -e \cdot -e \cdot -e \cdot -e \cdot -e = -e^5 = -e$ so that $(-e)^m = -e$ then e is m-potent element in $\mathcal{C}$ all odd positive integer number $m \geq 3$ therefor $-\kappa$ is mp-clean.

Proposition 2.16

If κ is mp-clean in $\mathcal{C}$ then $1 - \kappa$ is mp for each positive integer number $m \geq 2$.

Proof: Impose that κ is mp in $\mathcal{C}$ then $\kappa = e + p$ for some $p \in \mathcal{P}(\mathcal{C})$ e is m-potent element in $\mathcal{C}$ so $1 - \kappa = (1 - e) + (-p)$ we should prove $-p \in \mathcal{P}(\mathcal{C})$ and $(1 - e)^m = 1 - e$ is m-potent element in $\mathcal{C}$ for all positive integer number $m \geq 2$, since $p \in \mathcal{P}(\mathcal{C})$ so there exist b in $\mathcal{C}$ such s.t $p = p b$ so $-p = -p b = (-p)b$ so $-p \in \mathcal{P}(\mathcal{C})$ now since, if $m=2$ then $(1 - e)^2 = 1 - 2e + e^2$ since $e^2 = e$ so $(1 - e)^2 = 1 - e$ if $m=3$ then $(1 - e)^3 = (1 - 2e + e^2) \cdot (1 - e) = (1 - e) \cdot (1 - e) = 1 - e$ so that by the same way $(1 - e)^m = 1 - e$ is m-potent element in $\mathcal{C}$ for all positive integer number $m \geq 2$ therefore $1 - \kappa$ is mp-clean.

Proposition 2.17

If $\mathcal{C}_i$ be a family of rings. Such that $i=1 \dots k$ Then $\mathcal{C} = \mathcal{C}_1 \times \dots \times \mathcal{C}_k$ is *mp-clean* $\leftrightarrow \mathcal{C}_i$ is *mp-clean*.

Proof: Impose $\mathcal{C} = \mathcal{C}_1 \times \dots \times \mathcal{C}_k$ is *mp-clean* to prove that $\mathcal{C}_i$ is *mp-clean*, let $f: \mathcal{C} \rightarrow \mathcal{C}_i$ be a mapping so by proposition.2.9 homeomorphism of *mp-clean* is *mp-clean* so that $\mathcal{C}_i$ is *mp-clean*

Conversely: Suppose that $\mathcal{C}_i$ is *mp-clean* to prove that $\mathcal{C}$ is *mp-clean* we must show

that for every $\kappa = \kappa_i$ in $\mathbb{C}$ there exist $p = p_i$ in $P(\mathbb{C})$ and $e = e_i$ is m -potent such that $\kappa = p + e$ since $\mathbb{C}_i$ is $*mp$ -clean $*$ then $p = p_i$ pure elements in $\mathbb{C}$ and $e = e_i$ is m -potent in $\mathbb{C}$ so that κ is $*mp$ -clean $*$ element therefore $\mathbb{C}$ is $*mp$ -clean $*$.

Conclusion

In this paper we give some examples and properties of $*mp$ -clean ring $*$ and we study the connection between $*mp$ -clean ring $*$ and another rings (clean ring, m -clean ring, mj -clean ring, amalgamated ring and r -clean ring) such that clean ring, m -clean ring is strong than $*mp$ -clean ring $*$ also show that m -clean is $*mp$ -clean ring $*$ also amalgamated ring is $*mp$ -clean ring $*$ and prove that.

References

- [1] N. Ashrafi, E. Nasibi, *r-Clean Rings*, Math. Reports **15** (2013), 125-132.
- [2] W. Chen, S. Cui, *Notes on clean rings and clean elements*, Southeast Asian Bull. Math. **32** (2008), 855-861.
- [3] M. D'Anna, M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. **06** (2007), 443-459.
- [4] M. D'Anna, C. A. Finocchiaro, M. Fontana, *Amalgamated algebras along an ideal*, in: F. Marco, K. Salah-Eddine, O. Bruce, S. Irena (Eds.) Commutative Algebra and its Applications, De Gruyter, Berlin, New York, 2009, 155-172.
- [5] M. Esfandiar, H. Haj Seyyed Javadi, A. Moussavi, *A note on mj-clean rings*, Int. J. Nonlinear Anal. Appl. **15** (2024), 175-179.
- [6] A. S. Mohammed, I. S. Ahmed, S. H. Asaad, *Study of the rings in which each element express as the sum of an idempotent and pure*, Int. J. Nonlinear Anal. Appl. **12** (2021), 1719-1724.
- [7] W. K. Nicholson, *Lifting Idempotents and Exchange Rings*, Trans. Amer. Math. Soc. **229** (1977), 269-278.
- [8] S. Purkait, T. K. Dutta, S. Kar, *On m-clean and strongly m-clean rings*, Commun. Algebra **48** (2020), 218-227.
- [9] G. Ulucak, A. Kör, *On mj-clean ring and strongly mj-clean ring*, Turk. J. Math. **46** (2022), 2015-2022.
- [10] V. Vijayanand, C. Selvaraj, *BI-Amalgamated Rings with m-Clean Properties*, Far East J. Math. Sci. (FJMS) **140** (2023), 209-226.