

AN ADDITIONAL PROPERTIES FOR THE CATEGORIES OF UNPOINTED AND POINTED AND PAIRS SPACES

Abdulsatar Jmah Al-Juburie^{1*}

Abstract: Let $T_-(\circ)$, T_-^* be the category of unpointed and pointed spaces respectively. We construct and proof some additional properties for $T_-(\circ)$. Furthermore, we have construct and proof some propositions for both the category of T_-^* and the category of pairs by prove the analog of the results such as natural equivalence, evaluation map and exponential law of [1] in the categories of pointed spaces and pairs.

Introduction

Algebraic topology, a field of mathematics, utilizes methods from abstract algebra to investigate topological spaces [10, 23]. The central concept is to transform issues concerning topological spaces and continuous functions into issues concerning algebraic objects, as illustrated in [12], and [3]. Category theory adopts a bird's eye perspective on Mathematics [6, 15]. It offers a general theoretical basis that has shown to be advantageous in fields as varied as topology [18], geometry, and foundational mathematics. For instance, a topologist may work in the category of topological spaces and continuous functions, while an algebraist may work in the category of groups and homeomorphisms, as seen in [1], [7], [24], and [9]. Categories and functors are generally employed in the field of algebraic topology. The category of pointed spaces, unpointed spaces, pairs, and exponential law are significant objects in both category theory and algebraic topology, as noted in sources such as [12], [1], and [7]. Our work will concentrate on the characteristics of the categories of pointed spaces and pairs. In Section 2, we introduce the definitions of categories and functors, along with some results and properties that we will use. In Section 3, we describe the concept of unpointed spaces, outlining some useful properties and definitions relevant to our work, such as those in [21], [24], and . We have constructed and demonstrated some additional results concerning the category of unpointed spaces. In Section 4, we give a description of the categories of pointed spaces and pairs. Also, we give and prove some propositions which are related to these categories.

Preliminaries

In this section we will give the definitions of the categories, functors and homotopy. In addition, we give some results and properties that we will use them in our work,

¹ University of Diyala. ORCID iD: <https://orcid.org/8249-8644-0002-0000>

Email: abdulsatar.aljuburie@uodiyala.edu.iq

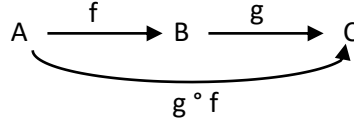
*Correspondence: abdulsatar.aljuburie@uodiyala.edu.iq

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further details may be found in [1, 3, 7, 12].

“Definition 2.1. A category $\mathcal{C}$ consists of a collection of objects $Ob(\mathcal{C})$ and for every $A, b \in Ob(\mathcal{C})$, a set $Mor_{\mathcal{C}}(A, B)$, called the set of morphisms (arrows) from A to B . These the set of following conditions”:

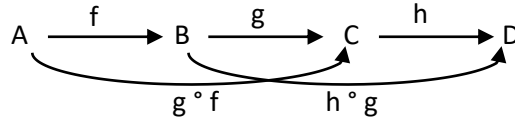
- i. “Composition: for any three objects $A, B, C \in Ob(\mathcal{C})$, for each pair of morphisms, $f \in mor_{\mathcal{C}}(A, B)$ and $g \in mor_{\mathcal{C}}(B, C)$, there exists a morphisms $g \circ f \in mor_{\mathcal{C}}(A, C)$. i.e.”



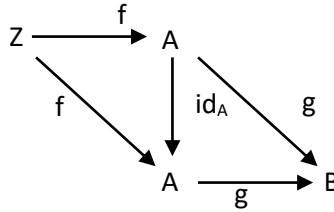
“Associativity: the composition operation is associative: $(h \circ g) \circ f = h \circ (g \circ f)$, i.e., if

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

is a sequence of morphisms, then we can compose them as,”



Identity: for any objects $A \in Ob(\mathcal{C})$ there exist $id_A \in Mor_{\mathcal{C}}(A, A)$ such, $id_A \circ f = f$ for $f \in Mor_{\mathcal{C}}(Z, A)$ and $g \circ id_A = g$ for $g \in Mor_{\mathcal{C}}(A, B)$. i.e.



Definition 2.2. Given two categories $\mathcal{C}$ and $\mathcal{D}$, a covariant functor $F : \mathcal{C} \rightarrow \mathcal{D}$ dose two things:[21]

- (i) associates to every $A \in Ob(\mathcal{C})$, an object $F(A) \in Ob(\mathcal{D})$ and
 (ii) for every $f \in Mor_{\mathcal{C}}(A, B)$, a morphism, $F(f) \in Mor_{\mathcal{D}}(F(A), F(B))$, [or : $F(f) \in Mor_{\mathcal{C}}(F(B), F(A))$ and this is a contravariant functor].”

“These must satisfy the following conditions:

- (iii) $F(g \circ f) = F(g) \circ F(f)$
 (iv) $F(id_A) = id_{F(A)}$ for every $A \in Ob(\mathcal{C})$.”

“Proposition 2.3 Boardman and Vogt [9] Let $\mathcal{C}$ be a category and let $A, B \in ob(\mathcal{C})$.

- (i) For $f: X \rightarrow Y$, write $f^*: mor_{\mathcal{C}}(Y, B) \rightarrow mor_{\mathcal{C}}(X, B)$ for the function $f^*: g \mapsto g \circ f$. Then the rules “

$F(X) = mor_{\mathcal{C}}(X, B)$ and $F(f) = f^*: F(Y) \rightarrow F(X)$ define a contravariant functor from $\mathcal{C}$ to Sets.

- (ii) For $f: X \rightarrow Y$, write $f_*: mor_{\mathcal{C}}(A, X) \rightarrow mor_{\mathcal{C}}(A, Y)$ for the function

$f_* : g \mapsto f \circ g$. Then the rules

$G(X) = \text{mor}_{\mathcal{C}}(A, X)$ and $G(f) = f_* : G(X) \rightarrow G(Y)$ define a covariant functor from $\mathcal{C}$ to Sets.

Definition 2.4 Given two functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$ (both covariant or both contravariant). A natural transformation is a rule Φ mapping:

$\Phi_X : F(X) \rightarrow G(X)$ for all objects $X \in \text{ob}(\mathcal{C})$ such that if $f : X \rightarrow Y$ is a morphism in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\Phi_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\Phi_Y} & G(Y) \end{array}$$

is commutes [20]

Definition 2.5. Let X, Y be a two space and I be the unite interval Two maps $f, g : X \rightarrow Y$ are called homotopic (written: $f \simeq g$) if there exists a continuous $\phi : X \times I \rightarrow Y$ such that $\phi(x, 0) = f(x)$ and $\phi(x, 1) = g(x)$ for each $x \in X$. If $g = \text{const}_y$ is a content map (i.e., there is $y \in Y$ such that $g(x) = y$ for all $x \in X$). Then we say that f is null-homotopic.

Theorem 2.6. The relation $f \simeq g$ is an equivalence relation on the set $\text{Hom}(X, Y)$ of continuous from X to Y . [26]

Proof: See [29].

Proposition 2.7 Carlsson [10] Let $f : A \rightarrow B$ be a morphism in the category $\mathcal{C}$. If the induce map $f_* : \text{mor}_{\mathcal{C}}(X, A) \rightarrow \text{mor}_{\mathcal{C}}(X, B)$ is bijection for every X , then f is an equivalence.

Proposition 2.7 Carlsson [10] Let $f : A \rightarrow B$ be a morphism in the category $\mathcal{C}$. If the induce map $f^* : \text{mor}_{\mathcal{C}}(B, X) \rightarrow \text{mor}_{\mathcal{C}}(A, X)$ is bijection for every X , then f is an equivalence."

Category of Unpointed Spaces

Let $T_*(\circ)$ denote the category of unpointed spaces, specifically the category of compactly generated Hausdorff spaces along with their continuous mappings. T^* denotes the category of pointed spaces. [4]

Definition 3.1 Grandis [16]: If X and Y are topological spaces, a mapping $F : X \rightarrow Y$ is called proclution if $f(X) = Y$ and a set U is open in Y whenever $f^{-1}(U)$ is open in X .

Theorem 3.2 Aguilar and Gitler [5] 4.4 If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are proclution as mapping in CG , the $f \times g : X \times Y \rightarrow X' \times Y'$ is also proclution in CG .

Defenition 3.3: Let X be a topological space, then X is called locally compact if for every open neighborhood W of x , there is an open neighborhood N with $\bar{N} \subset W$ and $\bar{U}$ is compact [2].

Definition 3.4: Assume that, X and Y are topological spaces. Let $\text{map}(X, Y)$ be the space of continuous mapping from $X \rightarrow Y$. Then the compact- open topology on $\text{map}(X, Y)$ is that: for each compact set $C \subset X$ and open set $N \subset Y$, let $S(C, N) = \{f \in \text{map}(X, Y) \mid f(C) \subset N\}$ set of all such that $S(C, N)$ is a subbasis for the compact -open topology on $\text{map}(X, Y)$. [13]

Definition 3.5 Suppose that X and Y are two topological spaces. Let $\text{map}(X, Y) = \{f : X \rightarrow Y \mid f \text{ is continuous}\}$ the set of all continuous maps from X to Y . The evaluation map is defined as follows, $@ : \text{map}(X, Y) \times X \rightarrow Y$ which is given by the

formula $(f, x) \mapsto f(x)$. [17]”

Lemma 3.6: Let T_0 be the category of unpointed spaces, which contains every locally compact space. Then the evaluation map $@: \text{map}(X, Y) \times X \rightarrow Y$ is continuous, for every $X, Y, Z \in T_0$. [11]

“Proof: Suppose that, $(f, x) \in \text{map}(X, Y) \times X$ and $N \subset Y$ a neighborhood of $@(f, x)$. We will show that, there exist a neighborhood of (f, x) that is mapped by $@$ into N . Now, because f is continuous and $f(x) \in N$, then $W = f^{-1}(N)$ is an open neighborhood of x in X . By the locally compactness property, that exist a neighborhood V of x such that $\bar{V} \subset W$ and $\bar{V}$ is compact. Note that, $V \times S(\bar{V}, N)$ is an open neighborhood of (f, x) since $f(\bar{V}) \subset N$. In addition, any $(f', x') \in V \times (\bar{V}, N)$ is mapped to N by $@$, because $x' \in V$ and f' takes V into N .”

Remark 3.7 The $\text{map}(X \times Y, Z) \cong \text{map}(X, \text{map}(Y, Z))$ is known as the exponential law.

Proposition 3.8 Awodey [8] Suppose that $A, X, Y \in T_0$ and $A \subseteq X$. Then $X/A \in T_0$ and $X \times Y \in T_0$.

Proposition 3.9 S. Li [22] Suppose that $F, G: T_0 \rightarrow T_0$ are two functors, defined by the formula:

$F(X) = \text{map}(X \times Y, Z)$ and $G(X) = \text{map}(X, \text{map}(Y, Z))$. Then the exponential law defines a natural equivalence between these two functors.

Proof: We define the two functors $F, G: T_0 \rightarrow T_0$ by the rules, $F(X) = \text{map}(X \times Y \times Z)$ and $G(X) = \text{map}(X, \text{map}(Y, Z))$. Let $\Phi: F(X) \rightarrow G(X)$. So, $\Phi: \text{map}(X \times Y, Z) \rightarrow \text{map}(X, \text{map}(Y, Z))$. We need to show that Φ defines a natural equivalence. Now, corresponding to an $f \in \text{map}(X \times Y, Z)$ then $f: X \times Y \rightarrow Z$ continuous such that $f((x, y)) = f_x(y)$. We define $\Phi f: X \rightarrow \text{map}(Y, Z)$ by $((\Phi f)_x)_y = f(x, y)$. To show this for each x , $(\Phi f)_x$ is continuous from Y to Z :

Suppose that $(\Phi f)_x$ takes $y_0 \in Y$ into the open set U of Z . Then $f(x, y_0) \in U$, now since f is continues, then this gives an open set V of Y containing y_0 such that, $f(X \times V) \subset U$. Therefore, $(\Phi f)_x$ map V into U . Now, we need to prove that:

If $f \in \text{map}(X, Y, Z)$, then $\Phi f: X \rightarrow \text{map}(Y, Z)$ is continuous.(*) [25]

Let $W(B, U)$ be the subbasis open set of $\text{map}(Y, Z)$, and suppose that, $(\Phi f)_{x_0} \in W(B, U)$ then $f(B \times x_0) \subset U$. Since, U is open and B is compact, then there is a neighborhood N of x_0 such that $f(B \times N) \subset U$. Implies that, $(\Phi f)_N \subset W(B, U)$ and we are done i.e., Φf is continuous. Now, to prove that Φ is continuous:

We will start with the continuity of the evaluation mapping by rearranged as,

$$@ : X \times Y \times \text{map}(X \times Y, Z) \rightarrow Z \text{ \{Cutler, 2020 \#25\}}$$

We will apply (*) with X replaced by $\times \text{map}(X \times Y, Z)$, then we will find that $\Phi @: X \times \text{map}(X \times Y, Z) \rightarrow \text{map}(Y, Z)$ is continuous. Now, we apply (*) again, with X replaced by $\text{map}(X \times Y, Z)$, Y by X and Z by $\text{map}(Y, Z)$, then we get that: $\Phi(\Phi @): \text{map}(X \times Y, Z) \rightarrow \text{map}(X, \text{map}(Y, Z))$ which is continuous. Note that, $\Phi(\Phi @)$ coincides Φ . We need to show that Φ has a continuous inverse:

Let $@: X \times \text{map}(X, \text{map}(Y, Z)) \rightarrow \text{map}(Y, Z)$ and $@': Y \times \text{map}(Z, Y) \rightarrow Z$ are evaluation mappings. Now, the composition $@': (1 \times @): Y \times X \times \text{map}(X, \text{map}(Y, Z)) \rightarrow Z$ is continuous (Theorem 4.3, part (C)), the composition map is continuous). [27]

Now, we applying $\odot$, with X replace by $\text{map}(X, \text{map}(Y, Z))$, Y by $Y \times X$, and Z by Z , we will get that:

$\Phi(@'(1 \times @))$: $\text{map}(X, \text{map}(Y, Z)) \rightarrow \text{map}(Y \times X, Z)$ is defined and continuous.

So, it is verified that $\Phi(@'(1 \times @))$ is the inverse of Φ . Hence, Φ is equivalence. [19]

Now we will use the exponential law to define a natural transformation:

Let $F, G : T_o \rightarrow T_o$ be two functors which are define by:

$$F(X) = \text{map}(X \times Y, Z) \text{ and } G(X) = \text{map}(X, \text{map}(Y, Z))$$

From above we get that,

$$\begin{aligned} \text{map}(X \times Y, Z) &\cong \text{map}(X, \text{map}(Y, Z)). \text{ So,} \\ F(X) = \text{map}(X \times Y, Z) &\cong \text{map}(X, \text{map}(Y, Z)) \end{aligned}$$

Let $g: X \rightarrow X'$ in T_o be any morphism. We know $\text{map}(X, Y) = \text{mor}_{Top}(X, Y) = \{f: X \rightarrow Y, \text{continuous}\}$. So we get the following diagram:

$$\begin{array}{ccc} F(X) = \text{map}_{Top}(X, \text{map}(Y, Z)) & \xrightarrow{\Phi_X = \phi_*} & G(X) = \text{map}_{Top}(X, \text{map}(Y, Z)) \\ F(g) = g^* \downarrow & & \downarrow G(g) = g^* \\ F(X') \text{map}_{Top}(X', \text{map}(Y, Z)) & \xrightarrow{\Phi_{X'} = \phi_*} & G(X') = \text{map}_{Top}(X', \text{map}(Y, Z)) \end{array}$$

Now, If $\Phi f \in \text{map}_{Top}(X, \text{map}(Y, Z)) \mapsto g^*(\Phi f) = (\phi f) \circ g \mapsto \phi_*((\phi f) \circ g) = \phi \circ (\Phi f \circ g)$.

Also, If $\Phi f \in \text{map}_{Top}(X, \text{map}(Y, Z)) \mapsto \phi_*(\Phi f) = \phi \circ (\Phi f) \mapsto g^*(\phi \circ (\Phi f)) = (\phi \circ (\Phi f)) \circ g = \phi \circ (\Phi f \circ g)$.

By Proposition 2.3 part (i), we get $F(g) = g^*$: $\Phi f \mapsto \Phi f \circ g$

Also, by Proposition 2.3 part (ii), we get $\Phi_X = \phi_*$: $\Phi f \mapsto \phi \circ \Phi f$

Similarly, we get: Turaev and Virelizier [28]

$$\Phi_Y = \phi_* : (\Phi f) \circ g \mapsto \phi \circ (\Phi f \circ g), \text{ and}$$

$$G(g) = g^*: \phi \circ \Phi f \mapsto (\phi \circ \Phi f) \circ g = \phi \circ (\Phi f \circ g) \text{ (associative).}$$

So, we get,

$$G(g) \circ \Phi_X : \Phi f \mapsto \phi \circ (\Phi f \circ g) \text{ and } \Phi_Y \circ F(g) : \Phi f \mapsto \phi \circ (\Phi f \circ g).$$

Hence, the diagram is commutative. Thus, $\Phi_X = \phi_* : F(X) \rightarrow G(X)$ is a natural transformation.

“Definition 3.10 Simmons [24] If X is a Hausdorff space, the associated compactly generated space $k(X)$ is the set X with topology defined as follows: a closed set of $k(X)$ is a set that meets each compact set of X in a closed set. If $f: X \rightarrow Y$ is a mapping of Hausdorff spaces, $k(f)$ denotes the same function $k(X) \rightarrow k(Y)$. k is the retraction functor.”

Theorem 3.11 May [20] If $f: X \rightarrow Y$ is continuous on compact sets, then $k(f)$ is continuous.

Definition 3.12 For Hausdorff spaces X, Y , define $\text{map}(X, Y) = k \text{map}(X, Y)$.

Theorem 3.13 Suppose that, X, Y , and Z in T_0 the composition of mappings $X \rightarrow Y \rightarrow Z$ is a continuous function $\text{map}(X, Z) \times \text{map}(X, Y) \rightarrow \text{map}(X, Z)$.

Proof: By Lemma 3.6, the mappings $\text{map}(X, Z) \times \text{map}(X, Y) \times X \xrightarrow{1 \times @} \text{map}(Y, Z) \times Y \xrightarrow{@'} Z$ are continuous, hence $@'(1 \times @)$ is also continuous. By applying $(*)$ in Proposition 4.9 with X replaced by $\text{map}(Y, Z) \times \text{map}(X, Y)$, Y by X, Z by Z and f by $@'(1 \times @)$, we will obtain that, $\phi(@'(1 \times @)) : \text{map}(Y, Z) \times \text{map}(X, Y) \rightarrow \text{map}(X, Z)$ is continuous. Hence, $k\phi(@'(1 \times @)) : \text{map}(Y, Z) \times \text{map}(X, Y) \rightarrow \text{map}(X, Z)$ is also continuous.

Category of Pairs and Category of Pointed Spaces

The category of pairs suppose that $A \subseteq X$ and $B \subseteq Y$. We consider only those maps $f: X \rightarrow Y$ with the additional property that $f(A) \subseteq B$. This type of map is written in the form $f: (X, A) \rightarrow (Y, B)$, and that (X, A) and (Y, B) are called pairs of spaces.

Definition 4.1: We denote by $\text{map}((X, A), (Y, B)) = \{f: (X, A) \rightarrow (Y, B)\} = \{f: X \rightarrow Y \mid f(A) \subseteq B\}$ the space of continuous mappings of pairs. It is clearly a subset of $\text{map}(X, Y)$; hence it inherits a topology from $\text{map}(X, Y)$. In fact, there is a category of pairs, which denote by $T_{(2)}$. There is an inclusion of categories $T_0 \rightarrow T_{(2)}$ given by $X \mapsto (X, \emptyset)$. The product of two pairs (X, A) and (Y, B) in T is also a pair in T :

$$(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B). [7]$$

A pointed space is a pair (X, x_0) , where x_0 is a point of X called the base point. We will use X_0 rather than (X, x_0) . The mapping $f: X_0 \rightarrow Y_0$ in T_* (the category of pointed spaces) is a mapping of pairs $f: (X, x_0) \rightarrow (Y, y_0)$. The smash product $X_0 \wedge Y_0$ is obtained from $X \times Y$ by collapsing the wedge $(X, y_0) \cup (x_0, Y)$ to a point that is the base point of $X_0 \wedge Y_0$. We define the function space of mappings of T_* by: $\text{map}(X_0, Y_0) = k\text{map}((X, x_0), (Y, y_0))$, such that the constant map $f_0(X) = y_0$ is the base point of this space. [11]

Now, our goal is to prove the analog of exponential $(*)$ in T_* . So, we need some preliminary result. Let $X \in CG$ and A be a closed subspace of X , where the space X/A obtained by collapsing A to a point a_0 (which is the base point) is a Hausdorff space. Suppose that, $\varphi: (X, A) \rightarrow (X/A)_0$ called the collapsing map. Let $Y_0 \in T_*$ belong to CG . Let $f: (X/A)_0 \rightarrow Y_0$. Note that,

$$\begin{array}{ccc} (X, A) & \xrightarrow{\varphi} & (X/A)_0 \xrightarrow{f} Y_0 \\ & \searrow \text{ } & \nearrow \text{ } \\ & f \circ \varphi & \end{array}$$

We will get that $f \circ \varphi \in \text{map}((X, A), (Y, y_0))$, and this will defines the following map of function spaces $\varphi^*: \text{map}((X/A)_0, Y_0) \rightarrow \text{map}((X, A), (Y, y_0))$. [14]

Proposition 4.2 φ^* is continuous and bijective, and it arrange a 1-1 correspondence between compact subsets. Therefore, by applying the functor k we get an induced natural equivalence $\text{map}((X/A)_0, Y_0) = \text{map}((X, A), (Y, y_0))$.

Proof: The continuity and bijective are directly proved. The important point is to prove that, if Φ is a compact subset of $\text{map}((X, A), (Y, y_0))$ then $\varphi^{*-1}(\Phi)$ is compact. Note that, it is enough to prove that φ^{*-1} is continuous on Φ . Suppose that, $h_0 \in \Phi$

and $W(M, U)$ is a subspaces open set of $\text{map}((X/A)_0, Y_0)$ containing $\varphi^{*-1} h_0$. This means that $h_0 \varphi$ maps M into U . Note that, in case M does not contain the base point a_0 , then $\varphi^{-1}(M)$ is compact in X and $W(\varphi^{-1}(M), U)$ is an open set that contains h_0 , and is mapped into $W(M, U)$ by φ^{*-1} . Therefore, Assume that M contains a_0 . Now, because Φ is compact that evaluation mapping $@: \Phi \times X \rightarrow Y$ is continuous, by Lemma 3.6. Note that, because $@(\Phi \times A) = y_0$ and $\Phi \times (X/A)$ is the decomposition space of $\Phi \times X$ obtained by collapsing $\Phi \times A$ to $\Phi \times a_0$ (see Theorem 3.2), it follows that $@$ induces a continuous mapping $@': \Phi \times (X/A) \rightarrow Y$. Because $@'(h_0, a_0) \in U$ then there exist a neighborhood V of h_0 in Φ and a neighborhood P of a_0 in X/A such that $@'$ maps $V \times p$ into U . Assume that $M' = M - M \cap P$, then M' is compact and dose not contain a_0 . This means $V \cap W(\varphi^{-1}M', U)$ is a neighborhood of h_0 in Φ and every $h \in V \cap W(\varphi^{-1}M', U)$ will maps $\varphi^{-1}P$ into U , since $h \in V$, and it will map $\varphi^{-1}M'$ into U since $h \in W(\varphi^{-1}M', U)$. Now, because $M \subset M' \cup P$ it follows that $\varphi^{*-1}h \in W(M, U)$. Hence, the proof of this proposition is completes.

Theorem 4.3 If X_0, Y_0, Z_0 are belong to T_* in the category of compactly generated CG , then $\text{map}(Y_0 \wedge X_0, Z_0) = \text{map}(X_0, \text{map}(Y_0, Z_0))$.

Proof: We will use W rather than $(Y \times x_0) \cup (y_0 \times X)$. As in Lemma 4.2, if we replace Y by Z and (X, A) by $(Y \times X, W)$, we will get the natural equivalence

$$\text{map}(Y_0 \wedge X_0, Z_0) = \text{map}((Y \times X, W), (Z, z_0))$$

Note that, the space $\text{map}((Y \times X, W), (Z, z_0))$ is a subspace of $\text{map}(Y \times X, Z)$ which by (Theorem 3.9) is equivalent te to $\text{map}(X, \text{map}(Y, Z))$, and the proof of this equivalent is the same proof in (Theorem 3.9). Now, $\text{map}((Y \times X, W), (Z, z_0))$ corresponds exactly to $\text{map}(X_0, \text{map}(Y_0, Z_0))$. This completes the proof.

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