

A NEW CHARACTERIZATION OF THE WEIGHTED COMPOSITION–DIFFERENTIATION OPERATORS ON BLOCH SPACES

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Abstract: This paper characterizes the boundedness and compactness of the weighted composition differentiation operator on the Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$ on the unit disk.

Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$ the class of all complex-valued functions analytic on $\mathbb{D}$. Let φ be an analytic self-map on $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. The weighted composition operator $W_{\psi,\varphi}$ induced by φ and ψ on $H(\mathbb{D})$ is defined by

$$W_{\psi,\varphi}f = \psi \cdot (f \circ \varphi), \quad f \in H(\mathbb{D}).$$

Weighted composition operators which appear in the theory of isometries on the spaces of analytic functions serve as a natural extension of both multiplication and composition operators. Moreover, this type of operator plays a role in the study of dynamical systems. For more information, we refer to the books [2, 15]. The differentiation operator $\mathcal{D}$ which is defined by $\mathcal{D}f = f'$ is typically unbounded on many analytic function spaces. The product of weighted composition operators and differentiation operators, denoted by $W_{\psi,\varphi}\mathcal{D}$, is defined as

$$W_{\psi,\varphi}\mathcal{D}f = \psi \cdot (f' \circ \varphi), \quad f \in H(\mathbb{D}).$$

We can consider a slightly broader class of operators, which are product of weighted composition operators and differentiation operators on many analytic function spaces and so define the *weighted composition–differentiation operator* $\mathcal{D}_{\psi,\varphi}$ by the rule

$$\mathcal{D}_{\psi,\varphi}f := \psi \cdot (f' \circ \varphi), \quad f \in H(\mathbb{D}).$$

To illustrate how this operator works, let's look at an example: Let $\psi(z) = 1$ and $\varphi(z) = z$. In this particular case, the operator $\mathcal{D}_{1,z}f(z) = f'(z)$. In this article, we consider the weighted composition–differentiation operator on the classical Bloch spaces.

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Background on the Bloch space

The classical Bloch space $\mathcal{B}(\mathbb{D})$ is the space of analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ satisfying

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty$$

endowed with the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)|$$

so that the Bloch space $(\mathcal{B}(\mathbb{D}), \|\cdot\|_{\mathcal{B}})$ becomes a Banach space. The little Bloch space $\mathcal{B}_0(\mathbb{D})$ consists of all $f \in \mathcal{B}(\mathbb{D})$ for which

$$\lim_{|z| \rightarrow 1} (1 - |z|^2) |f'(z)| = 0.$$

and $\mathcal{B}_0(\mathbb{D})$ is a closed subspace of $\mathcal{B}(\mathbb{D})$. For further information and references about the Bloch space $\mathcal{B}(\mathbb{D})$, the reader is referred to [13, 14, 19]. Note that $\mathcal{B} = \mathcal{B}(\mathbb{D})$ and $\mathcal{B}_0 = \mathcal{B}_0(\mathbb{D})$.

During recent years, there has been great interest in studying these types of operators between holomorphic spaces such as the Bergman space, the Hardy space, the weighted Banach spaces of holomorphic functions, and particularly the Bloch space, which plays an important role in geometric function theory.

For the special case $\psi(z) = 1$ and $\psi = \varphi'$, the operator $\mathcal{D}_{\psi, \varphi}$ reduce to the products of composition operator and differentiation operators $\mathcal{C}_{\varphi} \mathcal{D}$ and $\mathcal{D} \mathcal{C}_{\varphi}$ defined respectively by

$$\mathcal{C}_{\varphi} \mathcal{D} f = f'(\varphi) \text{ and } \mathcal{D} \mathcal{C}_{\varphi} f = f'(\varphi) \varphi'.$$

These operators were first characterized by Hibschiweiler and Portnoy in [3], and later by Ohno in [11], where the boundedness and compactness of operators of this type between Hardy spaces and Bergman spaces were investigated. In addition, Ohno studied these operators in [12], focusing on boundedness and compactness of operators $\mathcal{C}_{\varphi} \mathcal{D}$ and $\mathcal{D} \mathcal{C}_{\varphi}$ on the Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$. Furthermore, the compactness criterion for the linear product operator $\mathcal{D} \mathcal{C}_{\varphi}$ acting on the Bloch space was established in [18], in terms of the norms of φ^n .

It is specifically stated that in [16], Stević characterized the boundedness and compactness of weighted differentiation composition operators from the space of bounded analytic functions, the Bloch space and the little Bloch space to n th weighted-type spaces on the unit disk, and also in [17] the same author proved necessary and sufficient conditions for the weighted differentiation composition operator from the mixed-norm space to the weighted-type space or the little weighted-type space. In [9], Manhas and Zhao, who combined the weighted composition operator with the differentiation operator, characterized the boundedness and compactness of such operators between weighted Banach spaces of analytic functions with general weights. In [1], the authors gave some characterizations for the boundedness and compactness of weighted differentiation composition operator between different Bloch spaces. Finally, in [4], the authors characterized the boundedness and compactness, and provided asymptotic estimates of the norms of $W_{\psi, \varphi}$ in terms of the function theoretic properties of the symbol ψ and of the modulus $|\varphi_k|$ of a component function φ_k of the holomorphic symbol φ on (little) Bloch-type spaces.

For other studies on the characterization of various operators on spaces of holomorphic functions, see [5, 6, 10, 21] and, in particular [7, 20], which were the main sources of motivation for this paper.

The notation $a \lesssim b$ means that there is a positive constant C such that $a \leq Cb$. We say that $a \asymp b$, if both $a \lesssim b$ and $b \lesssim a$ hold.

Boundedness and compactness of $\mathcal{D}_{\psi,\varphi}$ on $\mathcal{B}$

In this section, we state and prove a boundedness and compactness criterion for the operator $\mathcal{D}_{\psi,\varphi}: \mathcal{B} \rightarrow \mathcal{B}$. For this purpose, we first quote some auxiliary results which will be used in the proofs of the main results in this paper. The following technical lemma was proved in [19].

Lemma 2.1. *Assume that $f \in \mathcal{B}$. Then for each $z \in \mathbb{D}$*

$$\|f\|_{\mathcal{B}} \asymp \sum_{j=0}^{n-1} |f^{(j)}(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^n |f^{(n)}(z)|.$$

As stated below, we characterize the boundedness of the weighted composition differentiation operator on the Bloch space in the first main theorem of the section. We begin with proving the boundedness criterion for $\mathcal{D}_{\psi,\varphi}$.

Theorem 2.2. *Let $\psi \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the operator $\mathcal{D}_{\psi,\varphi}: \mathcal{B} \rightarrow \mathcal{B}$ is bounded if and only if*

- i. $S_1 := \sup_{z \in \mathbb{D}} \frac{(1-|z|^2)|\psi(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^2} < \infty,$
- ii. $S_2 := \sup_{z \in \mathbb{D}} \frac{(1-|z|^2)|\psi'(z)|}{1-|\varphi(z)|^2} < \infty,$
- iii. $S_3 := \frac{|\psi(0)|}{1-|\varphi(0)|^2}.$

Moreover,

$$\|\mathcal{D}_{\psi,\varphi}\| \asymp S_1 + S_2 + S_3.$$

Proof. First, let's try to show that the operator $\mathcal{D}_{\psi,\varphi}$ is bounded using the conditions given by the theorem. Suppose that conditions (i) and (ii) hold. Let $f \in \mathcal{B}$. Using Lemma 2.1, we obtain the inequality

$$\begin{aligned} \|\psi \cdot (f' \circ \varphi)\|_{\mathcal{B}} &= |\psi(0)| |f'(\varphi(0))| + \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| (\psi \cdot (f' \circ \varphi))'(z) \right| \\ &= |\psi(0)| |f'(\varphi(0))| \\ &\quad + \sup_{z \in \mathbb{D}} (1 - |z|^2) |\psi'(z) f'(\varphi(z)) + \psi(z) f''(\varphi(z)) \varphi'(z)| \\ &\leq \left(\frac{|\psi(0)|}{1-|\varphi(0)|^2} + \sup_{z \in \mathbb{D}} \frac{(1-|z|^2)|\psi'(z)|}{1-|\varphi(z)|^2} + \sup_{z \in \mathbb{D}} \frac{(1-|z|^2)|\psi(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^2} \right) \|f\|_{\mathcal{B}} \\ &\leq (S_3 + S_2 + S_1) \|f\|_{\mathcal{B}}. \end{aligned}$$

Thus

$$\|\mathcal{D}_{\psi,\varphi}\| \leq S_1 + S_2 + S_3.$$

This proves that the operator $\mathcal{D}_{\psi,\varphi} : \mathcal{B} \rightarrow \mathcal{B}$ is bounded.

Now, under the assumption that the operator $\mathcal{D}_{\psi,\varphi}$ is bounded, we will prove that the stated conditions S_1 , S_2 and S_3 are satisfied using the chosen test functions $h_{\varphi(a)}, g_{\varphi(a)} \in \mathcal{B}$ for $a \in \mathbb{D}$. Suppose $\mathcal{D}_{\psi,\varphi} : \mathcal{B} \rightarrow \mathcal{B}$ is bounded, that is there exists a constant $C > 0$ such that

$$\|\mathcal{D}_{\psi,\varphi} f\|_{\mathcal{B}} \leq C \|f\|_{\mathcal{B}}$$

for all $f \in \mathcal{B}$. It is trivial that $\mathcal{D}_{\psi,\varphi} z = \psi \in \mathcal{B}$ satisfying $\|\psi\|_{\mathcal{B}} \leq C$. Also, by taking the function $f(z) = z^2$, using the facts that $\psi \in \mathcal{B}$ and that $\|\varphi\|_{\infty} \leq 1$, it follows that

$$\begin{aligned} \sup_{z \in \mathbb{D}} (1 - |z|^2) |\psi(z) \varphi'(z)| &\leq C + \left(\sup_{z \in \mathbb{D}} (1 - |z|^2) |\psi'(z)| \right) \left(\sup_{z \in \mathbb{D}} |\varphi(z)| \right) \\ &\leq 2C. \end{aligned} \quad (2.1)$$

Firstly, we prove that condition S_1 . We fix a point $a \in \mathbb{D}$ and consider the function

$$g_{\varphi(a)}(z) = \frac{2}{75} \frac{1 - |\varphi(a)|^2}{1 - \overline{\varphi(a)}z} - \frac{7}{300} \frac{(1 - |\varphi(a)|^2)^2}{(1 - \overline{\varphi(a)}z)^2} + \frac{1}{150} \frac{(1 - |\varphi(a)|^2)^3}{(1 - \overline{\varphi(a)}z)^3}.$$

Then $g_{\varphi(a)} \in \mathcal{B}$ for every $a \in \mathbb{D}$ satisfying $\sup_{a \in \mathbb{D}} \|g_{\varphi(a)}\|_{\mathcal{B}} \leq 1$. Also, since

$$g'_{\varphi(a)}(z) = \left[\frac{2}{75} \frac{1 - |\varphi(a)|^2}{(1 - \overline{\varphi(a)}z)^2} - \frac{7}{150} \frac{(1 - |\varphi(a)|^2)^2}{(1 - \overline{\varphi(a)}z)^3} + \frac{1}{50} \frac{(1 - |\varphi(a)|^2)^3}{(1 - \overline{\varphi(a)}z)^4} \right] \overline{\varphi(a)}$$

and

$$g''_{\varphi(a)}(z) = \left[\frac{4}{75} \frac{1 - |\varphi(a)|^2}{(1 - \overline{\varphi(a)}z)^3} - \frac{7}{50} \frac{(1 - |\varphi(a)|^2)^2}{(1 - \overline{\varphi(a)}z)^4} + \frac{2}{25} \frac{(1 - |\varphi(a)|^2)^3}{(1 - \overline{\varphi(a)}z)^5} \right] (\overline{\varphi(a)})^2,$$

we have $g'_{\varphi(a)}(\varphi(a)) = 0$ and $g''_{\varphi(a)}(\varphi(a)) = -\frac{1}{150} \frac{(\overline{\varphi(a)})^2}{(1 - |\varphi(a)|^2)^2}$ which implies that

$$\begin{aligned} \|\mathcal{D}_{\psi,\varphi}\| &= \sup_{\|f\|_{\mathcal{B}} \leq 1} \|\mathcal{D}_{\psi,\varphi} f\|_{\mathcal{B}} \\ &\geq \|\mathcal{D}_{\psi,\varphi} g_{\varphi(a)}\|_{\mathcal{B}} \\ &\geq \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| [\psi \cdot (g'_{\varphi(a)} \circ \varphi)]'(z) \right| \\ &\geq (1 - |a|^2) |\psi'(a) g'_{\varphi(a)}(\varphi(a)) + \psi(a) g''_{\varphi(a)}(\varphi(a)) \varphi'(a)| \\ &= (1 - |a|^2) \left| \frac{-\psi(a) \varphi'(a) (\overline{\varphi(a)})^2}{150(1 - |\varphi(a)|^2)^2} \right|. \end{aligned} \quad (2.2)$$

From (2.1) and (2.2), we have

$$\begin{aligned}
\sup_{z \in \mathbb{D}} \frac{(1-|z|^2)|\psi(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^2} &\leq \sup_{|\varphi(z)| > \frac{1}{2}} \frac{(1-|z|^2)|\psi(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^2} + \sup_{|\varphi(z)| \leq \frac{1}{2}} \frac{(1-|z|^2)|\psi(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^2} \\
&\leq \sup_{|\varphi(z)| > \frac{1}{2}} \frac{4(1-|z|^2)|\psi(z)\varphi'(z)\overline{(\varphi(z))}^2}{(1-|\varphi(z)|^2)^2} \\
&\quad + \sup_{|\varphi(z)| \leq \frac{1}{2}} \frac{16}{9}(1-|z|^2)|\psi(z)||\varphi'(z)|.
\end{aligned}$$

Thus, it follows that condition S_1 holds, as desired.

Secondly, in order to prove condition S_2 , for $a \in \mathbb{D}$, consider the function

$$h_{\varphi(a)}(z) = \frac{3}{86} \frac{1-|\varphi(a)|^2}{1-\overline{\varphi(a)}z} - \frac{1}{43} \frac{(1-|\varphi(a)|^2)^2}{(1-\overline{\varphi(a)}z)^2} + \frac{1}{172} \frac{(1-|\varphi(a)|^2)^3}{(1-\overline{\varphi(a)}z)^3}.$$

Then $h_{\varphi(a)} \in \mathcal{B}$ for every $a \in \mathbb{D}$ satisfying $\sup_{a \in \mathbb{D}} \|h_{\varphi(a)}\|_{\mathcal{B}} \leq 1$. Also, since

$$h'_{\varphi(a)}(z) = \left[\frac{3}{86} \frac{1-|\varphi(a)|^2}{(1-\overline{\varphi(a)}z)^2} - \frac{2}{43} \frac{(1-|\varphi(a)|^2)^2}{(1-\overline{\varphi(a)}z)^3} + \frac{3}{172} \frac{(1-|\varphi(a)|^2)^3}{(1-\overline{\varphi(a)}z)^4} \right] \overline{\varphi(a)}$$

and

$$h''_{\varphi(a)}(z) = \left[\frac{3}{43} \frac{1-|\varphi(a)|^2}{(1-\overline{\varphi(a)}z)^3} - \frac{6}{43} \frac{(1-|\varphi(a)|^2)^2}{(1-\overline{\varphi(a)}z)^4} + \frac{3}{43} \frac{(1-|\varphi(a)|^2)^3}{(1-\overline{\varphi(a)}z)^5} \right] (\overline{\varphi(a)})^2$$

we have $h'_{\varphi(a)}(\varphi(a)) = \frac{1}{172} \frac{\overline{\varphi(a)}}{1-|\varphi(a)|^2}$ and $h''_{\varphi(a)}(\varphi(a)) = 0$ which implies that

$$\begin{aligned}
\|\mathcal{D}_{\psi, \varphi}\| &= \sup_{\|f\|_{\mathcal{B}} \leq 1} \|\mathcal{D}_{\psi, \varphi} f\|_{\mathcal{B}} \\
&\geq \|\mathcal{D}_{\psi, \varphi} h_{\varphi(a)}\|_{\mathcal{B}} \\
&\geq \sup_{z \in \mathbb{D}} (1-|z|^2) \left| [\psi \cdot (h'_{\varphi(a)} \circ \varphi)]'(z) \right| \\
&\geq (1-|a|^2) |\psi'(a) h'_{\varphi(a)}(\varphi(a)) + \psi(a) h''_{\varphi(a)}(\varphi(a)) \varphi'(a)| \\
&= (1-|a|^2) \left| \frac{\psi'(a) \overline{\varphi(a)}}{172(1-|\varphi(a)|^2)} \right|.
\end{aligned} \tag{2.3}$$

From (2.3) and the facts that $\psi \in \mathcal{B}$ we have

$$\begin{aligned}
\sup_{z \in \mathbb{D}} \frac{(1-|z|^2)|\psi'(z)|}{1-|\varphi(z)|^2} &\leq \sup_{|\varphi(z)| > \frac{1}{2}} \frac{(1-|z|^2)|\psi'(z)|}{1-|\varphi(z)|^2} + \sup_{|\varphi(z)| \leq \frac{1}{2}} \frac{(1-|z|^2)|\psi'(z)|}{1-|\varphi(z)|^2} \\
&\leq \sup_{|\varphi(z)| > \frac{1}{2}} \frac{2(1-|z|^2)|\psi'(z)\overline{\varphi(z)}|}{1-|\varphi(z)|^2} + \frac{4}{3} \|\psi\|_{\mathcal{B}}.
\end{aligned}$$

Thus, it follows that condition S_2 holds, as desired.

Finally, condition S_3 is already finite. This completes the proof of the theorem.

In order to obtain the compactness of the operator $\mathcal{D}_{\psi,\varphi}$ we need the following lemma. The proof of Lemma is similar to the one for Proposition 3.11 in [2].

Lemma 2.3. *Let φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Then $\mathcal{D}_{\psi,\varphi}: \mathcal{B} \rightarrow \mathcal{B}$ is compact if and only if $\mathcal{D}_{\psi,\varphi}: \mathcal{B} \rightarrow \mathcal{B}$ is bounded and for any bounded sequence $\{f_n\}$ in $\mathcal{B}$ satisfying $f_n \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \rightarrow \infty$, we have $\|\mathcal{D}_{\psi,\varphi} f_n\|_{\mathcal{B}} \rightarrow 0$ as $n \rightarrow \infty$.*

In the second main theorem of the section, we characterize the compactness of the weighted composition differentiation operator on the Bloch space. In the proof of the theorem, Lemma 2.3 is used to establish the compactness of $\mathcal{D}_{\psi,\varphi}$, while the other aspect of the theorem is proved by contradiction. Also, the test functions $g_{\varphi(a)}, h_{\varphi(a)} \in \mathcal{B}$, ($a \in \mathbb{D}$) established in Theorem 2.2 also appear in the proof of this theorem.

Theorem 2.4. *Suppose φ is an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Then the following statements are equivalent*

- i. $\mathcal{D}_{\psi,\varphi}: \mathcal{B} \rightarrow \mathcal{B}$ is compact
- ii. $\mathcal{D}_{\psi,\varphi}: \mathcal{B} \rightarrow \mathcal{B}$ is bounded, and

$$\lim_{r \rightarrow 1} \sup_{|\varphi(z)| > r} \frac{(1-|z|^2)|\psi'(z)|}{1-|\varphi(z)|^2} = 0 \quad \text{and} \quad \lim_{r \rightarrow 1} \sup_{|\varphi(z)| > r} \frac{(1-|z|^2)|\psi(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^2} = 0. \quad (2.4)$$

Proof. Firstly, we assume that condition (ii) holds. To establish condition (i), let $\{f_n\}$ be a bounded sequence in $\mathcal{B}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$. To show that the operator $\mathcal{D}_{\psi,\varphi}$ is compact, according to Lemma 2.3, it is enough to show that $\|\mathcal{D}_{\psi,\varphi} f_n\|_{\mathcal{B}} \rightarrow 0$ as $n \rightarrow \infty$.

We may assume that $\|f_n\|_{\mathcal{B}} \leq 1$ for every $n \in \mathbb{N}$. In view of conditions (2.4), for every $\varepsilon > 0$, there exists $0 < r < 1$ such that when $r < |\varphi(z)| < 1$, we have

$$\frac{(1-|z|^2)|\psi'(z)|}{1-|\varphi(z)|^2} < \frac{\varepsilon}{5} \quad \text{and} \quad \frac{(1-|z|^2)|\psi(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^2} < \frac{\varepsilon}{5}. \quad (2.5)$$

Since $\mathcal{D}_{\psi,\varphi}$ is bounded, the conditions (i) and (ii) of Theorem 2.2 provide. Also, since $f_n \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimate gives that $f'_n \rightarrow 0$ and $f''_n \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$. This implies that there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{|\varphi(z)| \leq r} |f'_n(\varphi(z))| < \frac{\varepsilon}{5\|\psi\|_{\mathcal{B}}} \quad \text{and} \quad \sup_{|\varphi(z)| \leq r} |f''_n(\varphi(z))| < \frac{\varepsilon}{10c} \quad (2.6)$$

for every $n \geq n_0$.

Hence, using (2.5) and (2.6) we obtain that

$$\begin{aligned} \|\mathcal{D}_{\psi,\varphi} f_n\|_{\mathcal{B}} &\leq |\psi(0)||f'_n(\varphi(0))| + \sup_{z \in \mathbb{D}} (1-|z|^2) |\psi'(z)f'_n(\varphi(z)) + \psi(z)f''_n(\varphi(z))\varphi'(z)| \\ &\leq |\psi(0)||f'_n(\varphi(0))| \\ &\quad + \sup_{|\varphi(z)| \leq r} (1-|z|^2) |\psi'(z)f'_n(\varphi(z)) + \psi(z)f''_n(\varphi(z))\varphi'(z)| \\ &\quad + \sup_{|\varphi(z)| > r} (1-|z|^2) |\psi'(z)f'_n(\varphi(z)) + \psi(z)f''_n(\varphi(z))\varphi'(z)| \end{aligned}$$

$$\begin{aligned}
&\leq |\psi(0)| |f'_n(\varphi(0))| + \left(\sup_{|\varphi(z)| \leq r} (1 - |z|^2) |\psi'(z)| \right) \left(\sup_{|\varphi(z)| \leq r} |f'_n(\varphi(z))| \right) \\
&\quad + \left(\sup_{|\varphi(z)| \leq r} (1 - |z|^2) |\psi(z) \varphi'(z)| \right) \left(\sup_{|\varphi(z)| \leq r} |f''_n(\varphi(z))| \right) \\
&\quad + \sup_{|\varphi(z)| > r} \frac{(1 - |z|^2) |\psi'(z)|}{1 - |\varphi(z)|^2} \|f_n\|_{\mathcal{B}} + \sup_{|\varphi(z)| > r} \frac{(1 - |z|^2) |\psi(z) \varphi'(z)|}{(1 - |\varphi(z)|^2)^2} \|f_n\|_{\mathcal{B}} \\
&\leq \|\psi\|_{\mathcal{B}} \frac{\varepsilon}{5\|\psi\|_{\mathcal{B}}} + \|\psi\|_{\mathcal{B}} \frac{\varepsilon}{5\|\psi\|_{\mathcal{B}}} + 2C \frac{\varepsilon}{10C} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon
\end{aligned}$$

for every $n \geq n_0$. This proves that $\mathcal{D}_{\psi, \varphi} : \mathcal{B} \rightarrow \mathcal{B}$ is compact and hence the proof of the sufficiency part of theorem is complete.

Now we assume that (i) is true, that is, $\mathcal{D}_{\psi, \varphi} : \mathcal{B} \rightarrow \mathcal{B}$ is compact. Then it is clear that $\mathcal{D}_{\psi, \varphi} : \mathcal{B} \rightarrow \mathcal{B}$ is bounded.

To prove the first condition in (2.4) and obtain the contradiction, assume that

$$\lim_{r \rightarrow 1} \sup_{|\varphi(z)| > r} \frac{(1 - |z|^2) |\psi'(z)|}{1 - |\varphi(z)|^2} := \alpha > 0.$$

Then, for every sequence $\{r_n\} \subseteq (0, 1)$ with $r_n \rightarrow 1$ we have

$$\sup_{|\varphi(z)| > r_n} \frac{(1 - |z|^2) |\psi'(z)|}{1 - |\varphi(z)|^2} = \alpha > 0.$$

So we get a sequence $\{z_n\} \subseteq \mathbb{D}$ with $|\varphi(z_n)| \geq r_n$ such that $\lim_{n \rightarrow \infty} |\varphi(z_n)| = 1$. Consider the function

$$h_n(z) = \frac{3}{86} \frac{1 - |\varphi(z_n)|^2}{1 - \varphi(z_n)z} - \frac{1}{43} \frac{(1 - |\varphi(z_n)|^2)^2}{(1 - \varphi(z_n)z)^2} + \frac{1}{172} \frac{(1 - |\varphi(z_n)|^2)^3}{(1 - \varphi(z_n)z)^3}.$$

Hence we see that $h_n \in \mathcal{B}$ for every $n \in \mathbb{N}$ and the sequence is bounded with $\sup_{n \in \mathbb{N}} \|h_n\|_{\mathcal{B}} \leq 1$ moreover $h'_n(\varphi(z_n)) = \frac{1}{172} \frac{\overline{\varphi(z_n)}}{1 - |\varphi(z_n)|^2}$ and $h''_n(\varphi(z_n)) = 0$. Since $|\varphi(z_n)| \rightarrow 1$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \sup_{z \in K} |h_n(z)| = 0$$

for any compact subset K of $\mathbb{D}$. Thus $\{h_n\} \in \mathcal{B}$ is bounded and $h_n \xrightarrow{\tau_0} 0$. So by Lemma 2.3 we obtain that $\|\mathcal{D}_{\psi, \varphi} h_n\|_{\mathcal{B}} \rightarrow 0$ as $n \rightarrow \infty$. But on the other hand, we have

$$\begin{aligned}
\|\mathcal{D}_{\psi, \varphi} h_n\|_{\mathcal{B}} &\geq (1 - |z_n|^2) |\psi'(z_n) h'_n(\varphi(z_n)) + \psi(z_n) h''_n(\varphi(z_n)) \varphi'(z_n)| \\
&\geq \frac{1}{172} \frac{(1 - |z_n|^2) \psi'(z_n) |\overline{\varphi(z_n)}|}{1 - |\varphi(z_n)|^2} \\
&\geq \frac{\alpha}{172} |\overline{\varphi(z_n)}|
\end{aligned}$$

for every $n \in \mathbb{N}$. This leads to a contradiction.

To prove the second condition in (2.4) and obtain the contradiction, assume that

$$\lim_{r \rightarrow 1} \sup_{|\varphi(z)| > r} \frac{(1-|z|^2)|\psi(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^2} := \beta > 0.$$

Similarly we get a sequence $\{z_n\} \subseteq \mathbb{D}$ with $|\varphi(z_n)| \geq r_n$ such that $\lim_{n \rightarrow \infty} |\varphi(z_n)| = 1$.

Consider the function

$$g_n(z) = \frac{2}{75} \frac{1-|\varphi(z_n)|^2}{1-\overline{\varphi(z_n)}z} - \frac{7}{300} \frac{(1-|\varphi(z_n)|^2)^2}{(1-\overline{\varphi(z_n)}z)^2} + \frac{1}{150} \frac{(1-|\varphi(z_n)|^2)^3}{(1-\overline{\varphi(z_n)}z)^3}.$$

Hence we see that $g_n \in \mathcal{B}$ for every $n \in \mathbb{N}$ and the sequence is bounded with

$\sup_{n \in \mathbb{N}} \|g_n\|_{\mathcal{B}} \leq 1$ moreover $g_n''(\varphi(z_n)) = -\frac{1}{150} \frac{(\overline{\varphi(z_n)})^2}{(1-|\varphi(z_n)|^2)^2}$ and $g_n'(\varphi(z_n)) = 0$. Since $|\varphi(z_n)| \rightarrow 1$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \sup_{z \in K} |g_n(z)| = 0$$

for any compact subset K of $\mathbb{D}$. Thus $\{g_n\} \in \mathcal{B}$ is bounded and $g_n \xrightarrow{\tau_0} 0$. So by Lemma 2.3 we obtain that $\|\mathcal{D}_{\psi, \varphi} g_n\|_{\mathcal{B}} \rightarrow 0$ as $n \rightarrow \infty$. But on the other hand, we have

$$\begin{aligned} \|\mathcal{D}_{\psi, \varphi} g_n\|_{\mathcal{B}} &\geq (1-|z_n|^2) |\psi'(z_n) g_n'(\varphi(z_n)) + \psi(z_n) g_n''(\varphi(z_n)) \varphi'(z_n)| \\ &\geq \frac{1}{150} \frac{(1-|z_n|^2) |\psi(z_n)| |\varphi'(z_n)| |\overline{\varphi(z_n)}|^2}{(1-|\varphi(z_n)|^2)^2} \\ &\geq \frac{\beta}{150} |\overline{\varphi(z_n)}|^2 \end{aligned}$$

for every $n \in \mathbb{N}$. This also leads to a contradiction, hence the proof of the theorem is complete.

To characterize the compactness on $\mathcal{B}_0$, we will need the following result which is an modification of Lemma 1 proven in [8].

Lemma 2.5. *Suppose φ is an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Then $\mathcal{D}_{\psi, \varphi} : \mathcal{B}_0 \rightarrow \mathcal{B}_0$ is compact if and only if*

$$\lim_{|z| \rightarrow 1} \sup_{\|f\|_{\mathcal{B}} \leq 1} (1-|z|^2) |(\mathcal{D}_{\psi, \varphi} f)'(z)| = 0.$$

In another main theorem of the section, we characterize the compactness of the weighted composition differentiation operator on the little Bloch space. While one aspect of the proof of the theorem is based on Lemma 2.5, the other aspect is also proved by contradiction.

Theorem 2.6. *Suppose φ is an analytic self-map of $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. Then $\mathcal{D}_{\psi, \varphi} : \mathcal{B}_0 \rightarrow \mathcal{B}_0$ is compact if and only if*

$$\lim_{r \rightarrow 1} \sup_{|z| > r} \frac{(1-|z|^2) |\psi'(z)|}{1-|\varphi(z)|^2} = 0 \quad \text{and} \quad \lim_{r \rightarrow 1} \sup_{|z| > r} \frac{(1-|z|^2) |\psi(z)| |\varphi'(z)|}{(1-|\varphi(z)|^2)^2} = 0. \quad (2.7)$$

Proof. Suppose that both conditions in (2.7) hold. For $z \in \mathbb{D}$, we get

$$(1 - |z|^2) \left| (\mathcal{D}_{\psi, \varphi} f)'(z) \right| \leq \left(\frac{(1 - |z|^2) |\psi'(z)|}{1 - |\varphi(z)|^2} + \frac{(1 - |z|^2) |\psi(z)| |\varphi'(z)|}{(1 - |\varphi(z)|^2)^2} \right) \|f\|_{\mathcal{B}}.$$

Taking supremum over $f \in \mathcal{B}$ such that $\|f\|_{\mathcal{B}} \leq 1$ above inequality and then letting $|z| \rightarrow 1$, we obtain

$$\lim_{|z| \rightarrow 1} \sup_{\|f\|_{\mathcal{B}} \leq 1} (1 - |z|^2) \left| (\mathcal{D}_{\psi, \varphi} f)'(z) \right| = 0.$$

Hence, by Lemma 2.5 we see that $\mathcal{D}_{\psi, \varphi}$ is compact on $\mathcal{B}_0$.

Now suppose that $\mathcal{D}_{\psi, \varphi}$ is compact. Then $\mathcal{D}_{\psi, \varphi}$ is bounded. So, taking the function $f(z) = z$ we have $\mathcal{D}_{\psi, \varphi} z = \psi \in \mathcal{B}_0$ satisfying

$$\lim_{|z| \rightarrow 1} (1 - |z|^2) |\psi'(z)| = 0. \quad (2.8)$$

Also, if we take the function $f(z) = z^2$, then

$$(1 - |z|^2) |\psi(z) \varphi'(z)| \leq (1 - |z|^2) |\psi'(z)| |\varphi(z)| + (1 - |z|^2) |z|.$$

Using the facts that $\psi \in \mathcal{B}_0$ and $\|\varphi\|_{\infty} \leq 1$, we have

$$\lim_{|z| \rightarrow 1} (1 - |z|^2) |\psi(z) \varphi'(z)| = 0. \quad (2.9)$$

For now, we prove that

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2) |\psi'(z)|}{1 - |\varphi(z)|^2} = 0. \quad (2.10)$$

By the same reasoning used in the proof of Theorem 2.4 to obtain a contradiction, we can find a sequence $\{z_n\} \subseteq \mathbb{D}$ such that $\lim_{n \rightarrow \infty} |\varphi(z_n)| = 1$. Then consider the function $h_n(z)$ (as above) such that $h_n \in \mathcal{B}_0$ for every $n \in \mathbb{N}$ with $\sup_{n \in \mathbb{N}} \|h_n\|_{\mathcal{B}} \leq 1$ and also $\mathcal{D}_{\psi, \varphi} h_n \in \mathcal{B}_0$. Since $h_n \in \mathcal{B}_0$ is bounded and $h_n \xrightarrow{\tau_0} 0$, we obtain that $\|\mathcal{D}_{\psi, \varphi} h_n\|_{\mathcal{B}} \rightarrow 0$ as $n \rightarrow \infty$ and

$$\|\mathcal{D}_{\psi, \varphi} h_n\|_{\mathcal{B}} \geq \frac{1}{172} \frac{(1 - |z_n|^2) |\psi'(z_n)| |\overline{\varphi(z_n)}|}{1 - |\varphi(z_n)|^2} \geq \frac{\alpha}{172} |\overline{\varphi(z_n)}|$$

which is a contradiction. One can obtain with exactly the same idea that

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2) |\psi(z)| |\varphi'(z)|}{(1 - |\varphi(z)|^2)^2} = 0. \quad (2.11)$$

Lastly, we prove that (2.8) and (2.10) imply the first statement in (2.7). From (2.10), it follows that for every $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{(1-|z|^2)|\psi'(z)|}{1-|\varphi(z)|^2} < \varepsilon, \quad (2.12)$$

when $|\varphi(z)| > \delta$. Also using (2.8), it follows that $\gamma \in (0,1)$ such that

$$(1-|z|^2)|\psi'(z)| < \varepsilon(1-\delta^2) \quad (2.13)$$

when $|z| > \gamma$. Therefore, when $\gamma < |z| < 1$ and $\delta < |\varphi(z)| < 1$, by (2.12) we have

$$\frac{(1-|z|^2)|\psi'(z)|}{1-|\varphi(z)|^2} < \varepsilon. \quad (2.14)$$

On the other hand, when $\gamma < |z| < 1$ and $|\varphi(z)| \leq \delta < 1$, by (2.13) we obtain that

$$\frac{(1-|z|^2)|\psi'(z)|}{1-|\varphi(z)|^2} \leq \frac{(1-|z|^2)|\psi'(z)|}{1-\delta^2} < \varepsilon. \quad (2.15)$$

From (2.14) and (2.15), we obtain the first statement in (2.7), as desired. The proof of the second statement in (2.7) can be done in similar way. The proof is completed. In this last part, we will consider the boundedness and compactness of the operator $\mathcal{C}_\varphi \mathcal{D}$, which is the product of the composition operator and the derivative operator. We can show the following results. If $\psi(z) = 1$ for every $z \in \mathbb{D}$, then Theorem 2.2, Theorem 2.4 and Theorem 2.6 reduce to Theorem 3.1, Theorem 3.2 and Theorem 3.4 of [12], respectively.

Corollary 2.7. *Let φ be an analytic self-map of $\mathbb{D}$. Then the operator $\mathcal{C}_\varphi \mathcal{D} : \mathcal{B} \rightarrow \mathcal{B}$ is bounded if and only if*

$$\sup_{z \in \mathbb{D}} \frac{(1-|z|^2)|\varphi'(z)|}{(1-|\varphi(z)|^2)^2} < \infty.$$

Corollary 2.8. *Suppose φ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent*

- i. $\mathcal{C}_\varphi \mathcal{D} : \mathcal{B} \rightarrow \mathcal{B}$ is compact
- ii. $\mathcal{C}_\varphi \mathcal{D} : \mathcal{B} \rightarrow \mathcal{B}$ is bounded and

$$\lim_{r \rightarrow 1} \sup_{|\varphi(z)| > r} \frac{(1-|z|^2)|\varphi'(z)|}{(1-|\varphi(z)|^2)^2} = 0.$$

Corollary 2.9. *Suppose φ is an analytic self-map of $\mathbb{D}$. Then $\mathcal{C}_\varphi \mathcal{D} : \mathcal{B}_0 \rightarrow \mathcal{B}_0$ is compact if and only if*

$$\lim_{r \rightarrow 1} \sup_{|z| > r} \frac{(1-|z|^2)|\varphi'(z)|}{(1-|\varphi(z)|^2)^2} = 0.$$

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