

APPROXIMATING SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS NUMERICALLY USING EULER AND RUNGE–KUTTA METHODS

Ali Sami Abdullah^{1*}, Esraa Habeeb Khaleel²

Abstract: Differential equations are considered as the main applicable solution for different problems not only in mathematics but also in several science branches. Differential equations are used in many different fields; because of that, these equations become very important for implementing several numerical approximations. The Runge-Kutta Method is preferred to be used when it is desired to solve analytically complex models by applying basic mathematical techniques. Therefore, this work aims to present several examples, mainly numerical, showing how approximate solutions of certain differential equations with initial conditions can be obtained using the methodology of the random Runge-Kutta and Euler methods. The study also introduced a numerical approximation of differential equations with randomness, convergence, and its applications. The results show that the numerical values of the Variance ($\text{Var}(X_n)$) and Mean Square ($E(X_n)$) are closer to the exact or theoretical values of ($\text{Var}(X_n)$) and ($E(X_n)$). Also, the results show that the approximations that are obtained by the method of Runge Kutta in order four give good results compared to those established by the Euler method when the decrease occurs in the step parameter (h).

Introduction

Differential equations are a natural way of modeling or simulating real situations where there may be changes in one or more unknown functions with respect to variables that can be called independent [7]. Although they are real situations, these situations need a solution since they are usually problems that are posed and involve a single differential equation coupled to several functions that may or may not be unknown [13]. Stochastic differential equations are used in many different fields; because of that, these equations become very important for implementing several numerical approximations [18]. Generally, when there is a situation and these are transformed into equations, these equations are usually accompanied by an additional condition that manages to specify the state of a group of equations at a time or initial position. Together with a differential equation, they form what is known as initial value problems [25]. Integral differential equations implemented a novel comprehensive strategy for solving the issue of mechanical signal operation, which addressed the modeling of complex situations in biology in terms of digital

¹ Department of Mathematics, College of Computer Science and Mathematics, Tikrit University, Tikrit, Iraq.. ORCID id: <https://orcid.org/0009-0006-9790-9035>
Email: as230031pcm@st.tu.edu.iq

² Department of Mathematics, College of Computer Science and Mathematics, Tikrit University, Tikrit, Iraq.. ORCID id: <https://orcid.org/0009-0001-1104-7854>
Email: esraa.h.khaleel@tu.edu.iq

*Correspondence: as230031pcm@st.tu.edu.iq

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connection [16]. The differential equation is considered analytically as the mathematical resolution of equations through symbols, while qualitatively is the analysis and interpretation of the equation in generally graphic terms to understand the behavior of the simulated phenomenon. In fact, the differential equation can be considered numerically to be the study of equations through the use of the construction of computational algorithms to give more exact approximations [24]. Swiss physicist-mathematician Leonhard Paul Euler (1707-1783) was credited with the title of the leading mathematician of the 17th century and one of the greatest and most prolific of all time ways to interpret a differential equation [26]. Another study evaluated the application that may require applying the Euler method to solve the differential equation more deeply. Furthermore, the analysis of the numerical solution was implemented to accelerate the time steps for solving the differential equation by applying the Hybrid Node Loop. This will save the time cost for getting the precise results when the number of variables become very extensive [24]. Thus, the uncertainty in the numerical solution was considered in the Runge Kutta method, which is supported with several examples in terms of the time integral and extreme value with the expected results of the differential equation [29]. Generally, when there is a situation and these are transformed into equations, these equations are usually accompanied by an additional condition that manages to specify the state of a group of equations at a time or initial position. Also, Mean-squared, Variance, and numerical methods are extended and used in different subjects [22]. However, using Stochastic Taylor expansions, the Stochastic Taylor schemes can be obtained for handling the numerical solution [4]. It is important to understand that when applying Mathematics to real-world situations, one often encounters problems that cannot be solved analytically or exactly and whose solution must be addressed by numerical methods. For example, solving the problem related to the hydrodynamic phenomena in rivers would need certain of particular to applying models to this phenomenal [23]. Also, to solve different problems, such as analyzing an earthquake, given that its data are measurements taken in reality, random models should be close to the realistic to mimic that analytically. Thus, Euler's Method is used when the differential equations do not have an analytical solution. However, in real problems, this method is avoided because the error of the curves generated with respect to real behavior is greater than the error generated by other methods, and the step size of h is divided with the errors decrease [14]. It is a very simple method to implement but of low order, so depending on the degree of precision the user wants, h can be very small. One way to improve the method is to use a better approximation to the integer [28]. This can lead to understanding the advantages and disadvantages of Euler's method in a simple way that it has errors when the instantaneous slope changes rapidly within the X . For better approximations, not only the initial point should be considered, but an average of the start and end points; the problem is that the value of Y at that final point is not known [3].

In mathematics and computer science, Euler's method integrates numerical procedures to get the solution with initial boundary to the Ordinary Differential Equations (ODEs) that including the certain given information about the initial problem [1]. In fact, Euler's method can be considered as a one of the most simple method to solve a problems with an initial boundary values and simplest of the Runge–Kutta methods [12]. The global error would be proportional with the size of steps in the Euler method of the first order; on the other hand, the local error can be proportional to step size (the square of the step size). Euler's method is used to build complex method applications in engineering and science fields [11]. In many

applications, the difficulties in Taylor approximations lead to the determination of different derivatives. So, in the last years, the weak and strong approximation, that is established by the stochastic differential equations, are used to develop a numerical scheme on Taylor and Runge–Kutta type [20]. Euler's method approximates the solution function by means of a polygonal line. Only one solution to the problem can be obtained as a result [2]. An independent increase in the variable h is made. The use of this method is given if the interpretation of the equation cannot be given analytically [27]. It is the simplest method for solving DE and by using Euler's method, it is assumed that the hypothesis of Picard's Theorem is verified [10].

Around the year 1900, mathematicians C. Runge and M. W. Kutta's theories at each step of Euler's method moved along the tangent of a certain curve that is "close" to the unknown or sought curve [5]. The Runge-Kutta methods extend this geometric idea by using several derivatives or intermediate tangents instead of just one to approximate the unknown function. The simplest Runge-Kutta methods are obtained by using two intermediate derivatives [6]. The Runge-Kutta Method is preferred to be used when it is desired to solve analytically complex models by applying basic mathematical techniques (these numerical techniques are the basis for the solution and simulation of complex problems using computers) [17]. For example, in mechanical engineering, they are used to approximately solve special cases or applications of the Navier-Stokes equations, applying numerical techniques and subsequently solving them on a computer (these techniques are known as CFD or computational fluid dynamics). Especially the features of the Runge-Kutta method are a specialization of one-step numerical methods [21]. The differential equation can be considered numerically to be the study of equations through the use of the construction of computational algorithms to give more exact approximations. It replaces the initial value problem with the equivalent Integral. Although it has the local truncation error of the Taylor method, it does not require the calculation and evaluation of the function's derivatives. The Runge-Kutta method can be of second, third, or fourth order. It is part of the family of iterative methods, both implicit and explicit, to approximate the solutions for equations from the first order [9]. Therefore, in this work, the numerical construction of solutions for random differential equations from the Euler random scheme will be carried out, and some conditions for the convergence in the square mean to be stable will be demonstrated. The expected value and variance of the approximation process will be calculated.

Numerical Applications

This section uses some examples to explain how to approximate the solution and results at certain cases of Differential Equations (DE) based on the initial conditions of these equations.

Determination of Displacement by a Ground Movement

Suppose that the issue of defining the impact that may be on the ground of the earth during the situation of an earthquake disturbance. Suppose it starts at $t = 0$, and is $X(t) > 0, t \geq 0$, the horizon measure for the top to the bottom. Depending on the suggested applied linear model [19], the value of the relative displacement $X(t)$ is ruled with the following DE.

$$\begin{aligned}
& \ddot{X}(t) + 2\zeta\delta_0(X(t)) + 2\delta_0^2(X(t)) = -Y(t), (t \geq 0, \\
& X(0) = 0, \dot{X}(0) \\
& = 0)
\end{aligned} \tag{1}$$

where $Y(t)$ is the stochastic methodology process can be given by

$$Y(t) = \sum_{j=1}^m t a_j e^{-\alpha_j t} \cos(\delta_j(t) + \theta_j), \quad t > 0 \tag{2}$$

The root mean square solution of (1) has the form [8],

$$X(t) = -\left\{ \int_0^t (\hbar(t-z)Y(z))dz \right\} \tag{3}$$

with $\zeta < 1$, and whereas, the value of $\hbar(t)$ is equal to,

$$\hbar(t) = \frac{1}{\hat{\delta}_0} \{e^{-\xi\delta_0 t} \sin(\hat{\delta}_0(t))\}, \quad t \geq 0, \quad \hat{\delta}_0 = \delta_0 \sqrt{1 - \xi^2} \tag{4}$$

An expectation and autocorrelation function of a second-order value of stochastic process $Y(t)$ are,

$$E[Y(t)] = 0, \quad t \geq 0 \tag{5}$$

and

$$\Gamma_Y(u, v) = E[Y(u)Y(v)] = \frac{1}{2} \sum_{j=1}^m u v a_j^2 e^{-\alpha_j(u+v)} \cos((u-v)\delta_j), \quad u, v \geq 0 \tag{6}$$

Therefore, from Equations (3)-(6), so

$$E[X(t)] = 0, \text{ where } t \geq 0$$

and

$$\begin{aligned}
& \text{Var}[X(t)] = E[X(t)] \\
& = \iint_0^t \{ \hbar(t-u) (\hbar(t-v) (\Gamma_Y(u, v))) \} du dv, \text{ where } t \\
& \geq 0
\end{aligned} \tag{7}$$

For applying the Euler random scheme, a system of required DE in terms of the first order is obtained from an equation of order two [15]. So that,

$$X^1(t) = X(t), X^2(t) = \dot{X}(t) \text{ y } (X(t))^T = [X^1(t), X^2(t)]$$

The form of Eq. (1) is,

$$\dot{X}(t) = C(t)X(t) + \mathcal{F}(t)$$

Considering that $C(t)$ and $\mathcal{F}(t)$ have been obtained in

$$\begin{aligned} \vec{x} = X &= \begin{bmatrix} X^1 \\ X^2 \end{bmatrix}, & C(t) = C &= \begin{bmatrix} 0 & 1 \\ -\delta_0^2 & -2\delta_0\xi \end{bmatrix}, & \vec{\mathcal{F}}(t) &= \mathcal{F}(t) \\ &= \begin{bmatrix} 0 \\ -Y(t) \end{bmatrix} \end{aligned} \quad (8)$$

and $(X(0))^T = 0^T = [0,0]$. As $X_0^T = [0,0]$, the term for Euler's method here, represented by,

$$\begin{aligned} X_n &= (I + \hbar C)^n X_0 + \hbar \sum_{l=0}^{n-1} (I + \hbar C)^{n-l-1} \mathcal{F}(t_l) \\ &= \hbar \sum_{l=0}^{n-1} (I + \hbar C)^{n-l-1} \mathcal{F}(t_l) \end{aligned} \quad (9)$$

where the identity matrix is representing by I (size 2). Also, notes that mean square converges to the Euler method is guaranteed. Where the Lipschitz condition is obtained,

$$\|\mathbb{G}(X, t) - \mathbb{G}(Y, t)\|_{s \times r} \leq \|C\|_{\infty} \|X - Y\|_{s \times r} = S \|X - Y\|_{s \times r} \quad (10)$$

whereas, $S = \max\{1, \delta_0^2 + 2\delta_0(\xi)\}$, and $\int_{t_0}^{t_e} S d\tau < \infty$.

Considering that as $E[Y(t_l)] = 0$, obtained $E[X_n] = 0$ and $E[X_n](E[X_n])^T = 0$, and the covariance matrix of X_n is,

$$\begin{aligned} \Lambda_{X_n} &= E[X_n(X_n)^T] \\ &= \hbar^2 \sum_{l=0}^{n-1} \sum_{j=0}^{n-1} (I + \hbar C)^{n-l-1} \{E[\mathcal{F}(t_l)\{\mathcal{F}(t_l)^T\}]\} ((I \\ &+ \hbar(C))^{n-l-1})^T \} \end{aligned} \quad (11)$$

Being

$$E[\mathcal{F}(t_l)\{\mathcal{F}(t_l)^T\}] = \begin{bmatrix} 0 & 0 \\ 0 & E[\mathcal{F}(t_l)\{\mathcal{F}(t_l)\}] \end{bmatrix}$$

Whereas, $E[Y(t_l)(Y(t_j))]$ is defined by Eq (6). Considering $\Lambda_{X_n} = (V^{ij})_{2 \times 2}$ represent the square matrix shown by (11), V^{11} is an approximate variance of the second – order stochastic (1)

got from the Euler scalar random method. Figure 1 present $V^{11} = \text{Var}[X_n]$ to $\hbar$

$$= \left(V1 = \frac{1}{20}, V2 = \frac{1}{40}, \right.$$

and $V3$

$$= \frac{1}{80} \Big) \text{ where the variance } \text{Var}[X(t_n)] \text{ showed by Eq (7) in Eq (1), } \xi = 0.05 \text{ and } \delta_j = 1, a_j = (0.5)^j, 0 \leq j \leq 20 \text{ for } Y(t) \text{ in Eq. (2).}$$

This graph shows that the exact solution is the nearest when $\hbar$ reduction.

This behavior is works with the following,

$$\vec{X}_n \overrightarrow{n} \rightarrow \overrightarrow{\infty} \vec{X} \Rightarrow \Lambda_{\vec{X}_n} \overrightarrow{n} \rightarrow \overrightarrow{\infty} \Lambda_{\vec{X}} \quad (12)$$

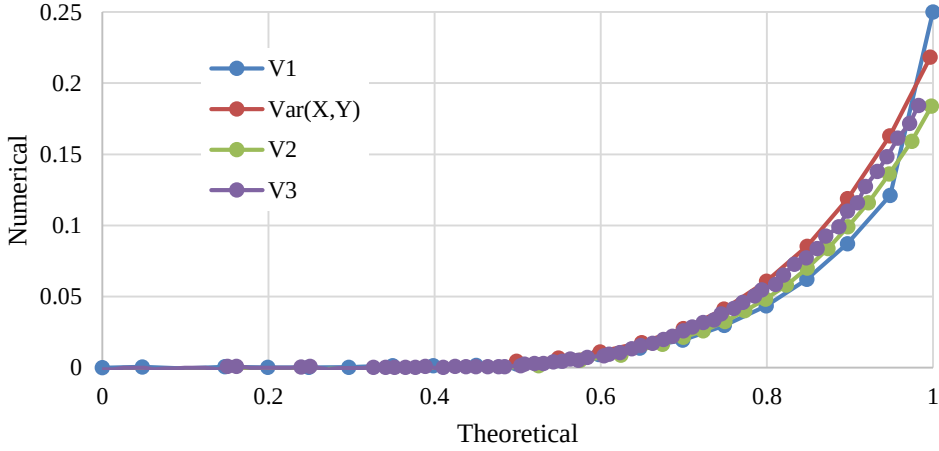


Figure 1: Comparison between Theoretical and Numerical Values.

Sylvester Complete Two-Sided Linear Matrix Problem

Suppose a system of non – homogeneous matrix random DE

$$\{\dot{X}(t)\} = C(t)(X(t)) + X(t)(Z(t)) + A(t), \quad \{t \geq 0 \text{ and } X(0) = X_0\} \quad (13)$$

whereas

$$C(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Z(t) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A(t) = \begin{bmatrix} Z(t) & 0 \\ 0 & (Z(t))^2 \end{bmatrix}, X_0 = 0, \quad (14)$$

where 0 is the null matrix and $Z(t)$ is a Brownian motion. Because of, $E[(Z(t))^4] = 3t^2$, as well as the value of $E[(Z(t))^2] = t$ therefore, when the range of domain is setting at $0 \leq t \leq t'$ this can be then approach the following,

$$\begin{aligned} & \|G(X, t) - G(X, t')\|_{s \times r} \\ &= \max \left\{ \|Z(t) - Z(t')\|, \left\| (Z(t))^2 - (Z(t'))^2 \right\| \right\} \max \left\{ |t - t'|^{1/2}, |-t^2 + 3(t')^2 - 2tt'|^{1/2} \right\} \end{aligned}$$

then the following will be obtained,

$$\begin{aligned} \|\mathbb{G}(X, t) - \mathbb{G}(X^*, t)\|_{s \times r} &\leq \|C(X - Y)\|_{s \times r} + \|(X - Y)Z\|_{s \times r} \\ &\leq (\|C\|_{\infty} + \|Z\|_{\infty})\|X - Y\|_{s \times r} \end{aligned}$$

Then Euler random method is convergent in the mean square. Denote $\{[X_n] = (E^{ij})_{2 \times 2}\}$, is an expectation of the matrix of the Euler method approximations solution. Figures (3 and 5) show the predicted result of different sets of values and steps. From these results, it is observed that the results obtained are coming closest as the value of decreases steps, as expected from the convergence in the square mean and, therefore, also of the mean and variance.

Linear Problem with Brownian Source Term

Suppose,

$$\{\dot{X}(t)\} = 2X(t) + e(-t) + Z(t), \quad (t \in [0, 1], \text{ and } X(t_0) = X_0) \quad (15)$$

where $Z(t)$ is the typical value of the Brownian process, X_0 is a random variable, $X_0 \sim N\left(\frac{1}{2}, \frac{1}{12}\right)$ independent of the Brownian motion $Z(t)$ for each value that satisfy the range $t \in [0, 1]$.

Now, to calculate the exact and practical solution, multiplying Eq.(14) by exponential $(-t^2)$ and, using $W(t) = \frac{dZ(t)}{dt}$, is obtained.

$$-2t(e(-t^2)(X(t))dt) + e(-t^2)dt = e(-t^2)((e(-t)) + Z(t))dt$$

Using Itô's formula (4), it follows.

$$\begin{aligned} d(e(-t^2)(X(t))) &= -2t(e(-t^2)(X(t))dt) + e(-t^2)dt \\ &= e(-t^2)((e(-t)) + Z(t))dt \end{aligned}$$

Hence

$$X(t) = e(t^2) \left\{ X_0 + \int_0^t e(-r^2) (e(-r) + Z(r))dr \right\} \quad (16)$$

Note that $F(X, t)$ satisfies the Lipschitz condition, given that

$$\begin{aligned} \|\mathbb{G}(X, t) - \mathbb{G}(X^*, t)\| &= \|2tX + \exp(-t) + Z(t) - 2tX^* - \exp(-t) - Z(t)\| \\ &= \|2tX - 2tX^*\| = 2t\|X - X^*\| \end{aligned}$$

and

$$\mathbb{G}(X, t) = 2tX + e(-t) + Z(t)$$

so,

$$\|\mathbb{G}(X, t) - \mathbb{G}(X^*, t)\| \leq (2\|X\| + 1)|t - t^*| + |t - t^*|^{1/2}$$

And $\mathbb{G}(X, t)$ is bounded randomly in a uniformly continuous way on a bounded group $S \subset L^2$. In this work, the Runge-Kutta of order 4 randomization theorem is

$$X_{n+1} = X_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \quad (17)$$

whereas,

$$K_1 = 2\hbar t_n X_n + \hbar(e(-t_n) + Z(t_n))$$

$$\begin{aligned} K_2 = & 2\hbar\left(t_n + \frac{h}{2}\right)(1 + \hbar t_n)X_n \\ & + \hbar^2\left(t_n + \frac{h}{2}\right)(e(-t_n) + Z(t_n)), + \hbar\left(e\left(-\left(t_n + \frac{h}{2}\right)\right)\right. \\ & \left.+ Z\left(t_n + \frac{h}{2}\right)\right) \end{aligned}$$

$$\begin{aligned} K_3 = & 2\hbar\left(t_n + \frac{h}{2}\right)(1 + \hbar\left(t_n + \frac{h}{2}\right)(1 + \hbar t_n))X_n + \hbar^3\left(t_n + \frac{h}{2}\right)^2(e(-t_n) + Z(t_n)) \\ & + \hbar(1 + \hbar\left(t_n + \frac{h}{2}\right))\left(e\left(-\left(t_n + \frac{h}{2}\right)\right) + Z\left(t_n + \frac{h}{2}\right)\right) \end{aligned}$$

$$\begin{aligned} K_4 = & 2\hbar(t_n + \hbar)(1 + 2\hbar\left(t_n + \frac{h}{2}\right)\left(1 + \hbar\left(t_n + \frac{h}{2}\right)(1 + \hbar t_n)\right)X_n + 2\hbar^4(t_n + \hbar) \\ & \times \left(t_n + \frac{h}{2}\right)^2(e(-t_n) + Z(t_n))2\hbar^2(t_n + \hbar)\left(1 + \hbar\left(t_n + \frac{h}{2}\right)\right) \\ & \times \left(e\left(-\left(t_n + \frac{h}{2}\right)\right) + Z\left(t_n + \frac{h}{2}\right)\right) \\ & + h(e(-(t_n + \hbar)) + Z(t_n + \hbar)) \end{aligned}$$

and denoting

$$\begin{aligned} a_n = & 1 + \frac{\hbar}{3}t_n + \frac{2\hbar}{3}\left(t_n + \frac{h}{2}\right)\left(1 + (1 + \hbar t_n)\left(1 + \hbar\left(t_n + \frac{h}{2}\right)\right)\right) \\ & + \frac{\hbar}{3}(t_n + \hbar)\left(1 + 2\hbar\left(t_n + \frac{h}{2}\right)\left(1 + \hbar\left(t_n + \frac{h}{2}\right)(1 + \hbar t_n)\right)\right) \end{aligned}$$

and

$$\begin{aligned}
 b_n = & \frac{h}{6} \left(1 + 2h \left(t_n + \frac{h}{2} \right) + 2h^2 \left(t_n + \frac{h}{2} \right)^2 + 2h^3 (t_n + h) \left(t_n + \frac{h}{2} \right)^2 \right) (e(-t_n)) \\
 & + Z(t_n) + \frac{h}{3} \left(2 + h \left(t_n + \frac{h}{2} \right) + h(t_n + h) \left(1 + h \left(t_n + \frac{h}{2} \right) \right) \right) \\
 & \times \left(e \left(- \left(t_n + \frac{h}{2} \right) \right) + Z \left(t_n + \frac{h}{2} \right) \right) \\
 & + \frac{h}{6} (e(-(t_n + h)) + Z(t_n + h))
 \end{aligned}$$

have

$$X_{n+1} = a_n X_n + b_n, \quad n = 0, 1, 2, \dots \quad (18)$$

hence

$$X_n = \left(\prod_{l=0}^{n-1} a_l \right) X_0 + \sum_{l=0}^{n-1} \left(\prod_{j=l+1}^{n-1} a_j \right) b_l \quad (19)$$

From Eq.(14) and Eq. (19), the expectation and variance of $X(t)$ and X_n are obtained,

$$E[X(t)] = e(t^2) \left[\frac{1}{2} + \int_0^t e(-r^2 - r) dr \right] \quad (20)$$

$$E[X_n] = \frac{1}{2} \prod_{l=0}^{n-1} a_l + \sum_{l=0}^{n-1} \left(\prod_{j=l+1}^{n-1} a_j \right) E[b_l] \quad (21)$$

whereas,

$$\begin{aligned}
 E[b_l] = & \frac{h}{6} \left(1 + 2h \left(t_l + \frac{h}{2} \right) + 2h^2 \left(t_l + \frac{h}{2} \right)^2 + 2h^3 (t_l + h) \left(t_l + \frac{h}{2} \right)^2 \right) e(-t_l) \\
 & + \frac{h}{3} \left(2 + h \left(t_l + \frac{h}{2} \right) \right. \\
 & \left. + h(t_l + h) \left(1 + h \left(t_l + \frac{h}{2} \right) \right) \right) \left(e \left(- \left(t_l + \frac{h}{2} \right) \right) \right) \\
 & + \frac{h}{6} (e(-(t_n + h)))
 \end{aligned}$$

and,

$$\begin{aligned}
Var[X(t)] &= e(2t^2) \left[\frac{1}{12} + \int_0^t \int_0^t e(-r^2 - s^2) \min(r, s) dr ds \right] \\
&= e(2t^2) \left[\frac{1}{12} + \int_0^t e(-r^2) \right. \\
&\quad \left. - e(-2r^2) \right) dr] \tag{22}
\end{aligned}$$

$$Var[X_n] = \frac{1}{2} \left(\prod_{l=0}^{n-1} a_l \right)^2 + \sum_{l=0}^{n-1} \sum_{s=0}^{n-1} \left(\prod_{j=l+1}^{n-1} a_j \right) \left(\prod_{l=s+1}^{n-1} a_l \right) Cov[b_l, b_s] \tag{23}$$

whereas,

$$\begin{aligned}
Cov[b_l, b_s] &= A_{l,s} \min(t_l, t_s) \\
&+ B_{l,k} \min\left(t_l, t_s + \frac{h}{2}\right) + C_l \min(t_l, t_s + h) + B_{s,l} \min\left(t_l + \frac{h}{2}, t_s\right) \\
&+ D_{s,l} \min\left(t_l + \frac{h}{2}, t_s + \frac{h}{2}\right) + E_l \min\left(t_l + \frac{h}{2}, t_s + h\right) \\
&+ C_s \min(t_l + h, t_s) \\
&+ E_s \min\left(t_l + h, t_s + \frac{h}{2}\right) + \frac{h^2}{36} \min(t_l + h, t_s + h)
\end{aligned}$$

being,

$$\begin{aligned}
C_{l,s} &= \frac{h^2}{36} \left(1 + 2h \left(t_l + \frac{h}{2} \right) + 2h^2 \left(t_l + \frac{h}{2} \right)^2 + 2h^3 (t_l + h) \left(t_l + \frac{h}{2} \right)^2 \right), \\
&\quad \times \left(1 + 2h \left(t_s + \frac{h}{2} \right) + 2h^2 \left(t_s + \frac{h}{2} \right)^2 + 2h^3 (t_s + h) \left(t_s + \frac{h}{2} \right)^2 \right) \\
Z_{l,c} &= \frac{h^2}{18} \left(1 + 2h \left(t_l + \frac{h}{2} \right) + 2h^2 \left(t_l + \frac{h}{2} \right)^2 + 2h^3 (t_l + h) \left(t_l + \frac{h}{2} \right)^2 \right), \\
&\quad \times \left(2 + h \left(t_c + \frac{h}{2} \right) + h(t_c + h)(1 + h \left(t_c + \frac{h}{2} \right)) \right) \\
A_l &= \frac{h^2}{36} \left(1 + 2h \left(t_l + \frac{h}{2} \right) + 2h^2 \left(t_l + \frac{h}{2} \right)^2 + 2h^3 (t_l + h) \left(t_l + \frac{h}{2} \right)^2 \right) \\
D_{l,c} &= \frac{h^2}{9} \left(2 + h \left(t_l + \frac{h}{2} \right) + h(t_l + h) \left(1 + h \left(t_l + \frac{h}{2} \right) \right) \right), \\
&\quad \times \left(2 + h \left(t_c + \frac{h}{2} \right) + h(t_c + h)^2 + \left(1 + h \left(t_c + \frac{h}{2} \right) \right) \right) \\
E_l &= \frac{h^2}{18} \left(2 + h \left(t_l + \frac{h}{2} \right) + h(t_l + h) \left(1 + h \left(t_l + \frac{h}{2} \right) \right) \right)
\end{aligned}$$

The value of the error of the variance and expectation of $X(t)$ with the Euler and Runge-Kutta methods of 4th order, in $h = (\frac{1}{20}, \frac{1}{40})$, are presented in Table 1. We have represented, for comparison, the expectation and variance, respectively, of Example 2.3 using the Runge-Kutta method of order 4. Figures (2, 3, 4, and 5) afford a clear picture of the range of values resulting from the implementation of this matrix under different ranges of the substituted step h . From the results, it is observed that the set of the obtained numerical results is closer to the value set of step h when it decreases, as expected by the convergence in square mean and, therefore, also of the mean and the variance. Furthermore, Table 1 shows that the numerical values of $E(X_n)$ and $Var(X_n)$ are closer to the theoretical values of $E(X_n)$ and $Var(X_n)$. Also, the results show that the approximations that are obtained by the method of Runge Kutta in order four give good results compared to those established by the Euler method when the decrease occurs in the step parameter (h).

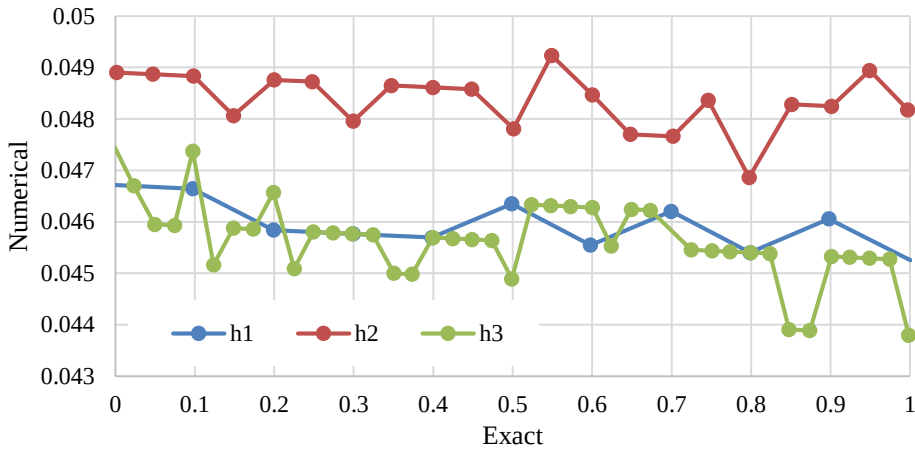


Figure 2: Numerical Values of E^{11} of the Expectation Matrix.

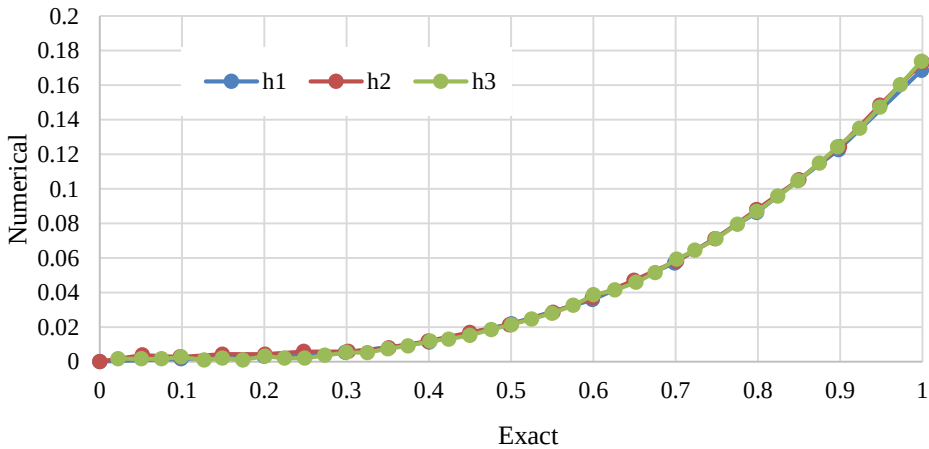


Figure 3: Numerical Values of E^{12} of the Expectation Matrix.

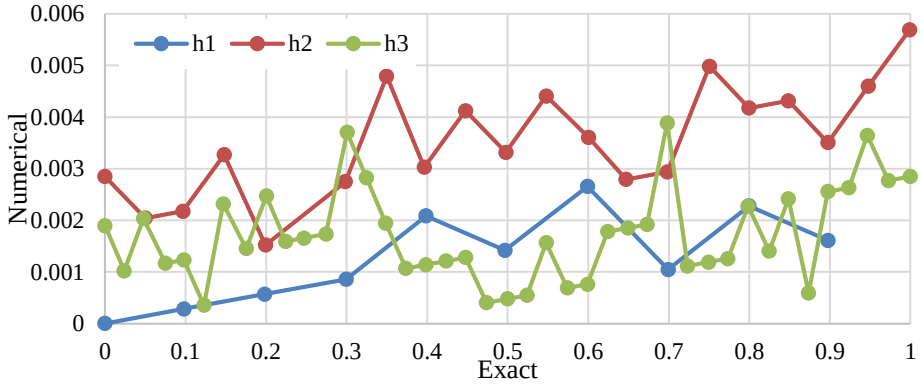


Figure 4: Numerical Values of E^{21} of the Expectation Matrix.

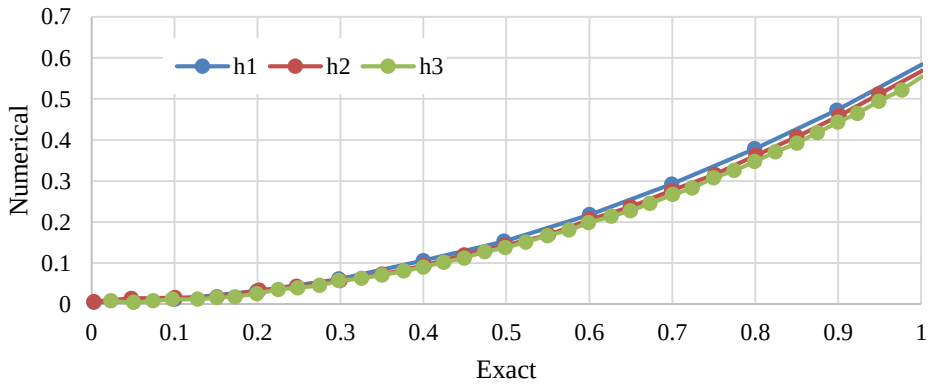


Figure 5: Numerical Values of E^{22} of the Expectation Matrix.

Table 1: The error value of the variance and expectation of $X(t)$ to Runge-Kutta and Euler of 4th order of the Brownian Problem 2.3.

t	$E(X_t)$				$Var(X_t)$			
	Runge-Kutta		Euler Method		Runge-Kutta		Euler Method	
	$h = \frac{1}{20}$	$h = \frac{1}{40}$	$h = \frac{1}{20}$	$h = \frac{1}{40}$	$h = \frac{1}{20}$	$h = \frac{1}{40}$	$h = \frac{1}{20}$	$h = \frac{1}{40}$
0.1	0.00000239	0.0000000484	0.000115	0.000347	0.00000654	0.00000214	0.00321	0.00279
0.2	0.00000166	0.000000375	0.00135	0.000876	0.0000247	0.00000325	0.00921	0.00952
0.3	0.00000486	0.000000786	0.00396	0.00329	0.0000335	0.00000427	0.0476	0.0327
0.4	0.0000105	0.00000264	0.00736	0.00538	0.0000457	0.00000675	0.0645	0.0637
0.5	0.0000272	0.00000374	0.0247	0.00875	0.0000598	0.00000884	0.210	0.204
0.6	0.0000362	0.00000426	0.0338	0.0325	0.0000797	0.0000210	0.217	0.287
0.7	0.0000468	0.00000685	0.0498	0.0213	0.0000978	0.0000267	0.415	0.446
0.8	0.0000591	0.00000878	0.0657	0.0322	0.000247	0.0000342	0.653	0.674
0.9	0.0000639	0.00000893	0.0958	0.0564	0.000365	0.0000428	0.985	0.964
1	0.0000887	0.0000134	0.149	0.0864	0.000415	0.0000579	1.8635	1.896

Conclusions

Differential equations are used in many different fields; because of that, these equations become very important for implementing several numerical

approximations. In this research, the analytical approximation solution for the differential equations under randomness, with the study of its convergence and its application, has been introduced. Using the Runge-Kutta and Euler method has allowed us to focus in order to deal with some examples where types of equations with regular sample behavior appeared. However, in some cases of equations, which are very irregular as they are not differential at any point, they lead to other types of differential equations with uncertainty called stochastic differential equations, whose approximation is addressed with alternative techniques with higher order methods.

From both methods, the results established by the Runge-Kutta method in the fourth-order can be considered as a best choice to provide a small margins in terms of error compared to the real result value of the problem. It also has the advantage of being easily programmable in software to perform the necessary iterations.

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