

Studying Some Fundamental Properties of Fuzzy Semi –Normal Spaces

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Abstract: The many kinds of fuzzy semi-normal spaces that are covered in detail in this chapter include fuzzy semi-star normal space (f.s*.normal space), fuzzy semi-normal space (f.s.normal space), and fuzzy semi-general normal space (f.sg.normal space). We use fuzzy semi-general open sets (f.sg-open set) to study the basic features of these spaces and to examine the interrelationships between them under certain constraints. We also present and analyze some more fuzzy spaces, including the S-regular space, the fuzzy semi-T_{1/2} space, and the semi-symmetric space. These spaces are examined in light of the fuzzy semi-normal spaces that were previously covered, emphasizing how they are related and, in certain cases, dependent on one another. The goal of this extensive study is to improve fuzzy topology theory and lay the groundwork for further investigations and real-world applications utilizing intricate fuzzy systems.

Introduction

Studying fuzzy semi-normal spaces is a complex and rich area within fuzzy topology that connects theoretical ideas with real-world applications [1, 3, 17]. These spaces are pivotal in understanding the broader framework of fuzzy topological structures [7, 9, 19]. Our investigation delves into the fundamental properties of these spaces through the lens of fuzzy semi-general open sets (f.sg-open sets) [2, 11, 17]. Fuzzy semi-star normal space (f.s*.normal space), fuzzy semi-normal space (f.s.normal space), and fuzzy semi-general normal space (f.sg.normal space) are some of the categories of fuzzy semi-normal spaces that are thoroughly examined in this chapter [5]. These sets are an essential tool for clarifying the complex hierarchies and interactions between various fuzzy spaces [6, 16, 18]. Our goal is to identify the underlying principles that drive these relationships' behavior and interconnectivity by investigating them under particular settings [8, 10, 15]. More fuzzy spaces, like S-regular space, fuzzy semi-T_{1/2} space, and semi-symmetric space, are also introduced [12, 14]. The fuzzy semi-normal spaces are examined in connection to these spaces in order to give a thorough knowledge of their interdependencies and the circumstances in which they function [4, 13].

Definition 2.1

For every fuzzy closed G and H in (A, T) , there exists an f.t.s (A, T) such that $\min \{\mu$

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$G(x), \mu H(x)\} = 0$ f.s.o.s. V and U in (A, T) such that $\mu G(x) \leq \mu V(x), \mu H \leq \mu U(x)$ and $\min \{\mu V(x), \mu U(x)\} = 0$.

Theorem 2.2

Every f.normal is a f. S. Normal

Proof:

Depending on the proposition (1.3.3)

Remark 2.3

As demonstrated by the following example, the claim (2.2)'s opposite need not be true:

Example 2.4

Let $X = \{(a, 0.7), (b, 0.8), (c, 0.4), (d, 0.6)\}$ and let $T_x = \{X, \emptyset, \{(a, 0.7), (b, 0.0), (c, 0.0), (d, 0.0)\}, \{(a, 0.0), (b, 0.8), (c, 0.0), (d, 0.0)\}, \{(a, 0.7), (b, 0.8), \{(c, 0.0), (d, 0.0)\}, \{(a, 0.7), (b, 0.8), (c, 0.4), (d, 0.0)\}, \{(a, 0.7), (b, 0.8), (c, 0.0), (d, 0.6)\}\}$ (X, T) is a f. Normal but is not f.s.normal.s

Definition 2.5

If there is a f.o.s. for each f.s.o.s. in (A, T) , then the fuzzy topological space (A, T) is called f.s. space.

Example 2.6

$X = \{(c, 0.8), (d, 0.8)\}$ and $T = \{X, \emptyset, \{(C, 0.7), (D, 0.0)\}\}$
Clearly (X, T) is a f.s.space.

Proposition 2.7

If (A, T) is a f.t.s., then the following claims are interchangeable:

- 1) (A, T) is a f.s.space .
- 2) Every f.s.c.s is a f.c.s.

Proof

It is clear

The following theorem shows that when (A, T) is a f.s.normal .s and f.s.pace then (A, T) is f.normal.space :-

Theorem 2.8

Given a f.s.normal space (A, T) , then (A, T) is a f.normal space if and only if (A, T) is a f.s.normal space.

Proof

Let (A, T) be a f.s.normal . space, f.s.space, let G and H be f.c.s. in (A, T) such that $\min \{ \mu G(x), \mu H(x) \} = 0$.

such that $\mu G(x) \leq \mu U(x), \mu H(x) \leq \mu V(x)$

Now, since (A, T) is a f.s.space then U and V are f.o.s in (A, T) .

Therefore (A, T) is a f.normal.space.

Now we are going to introduce a fuzzy semi – regular space (f.s.regulare.s) and we shall explain later that a f.s.normal.space need not be a f.s.regular.space unless it is a f.T1 – space :-

Definition 2.9 [1].

If there are two f.s.o.s., U and V in (A, T) such that $\Pr \leq \mu V$ and $\mu H(x) < \mu U(x)$, $\min \{ \mu U(x), \mu V(x) \} = 0$, then every a f.c.s. H of (A, T) and a fuzzy point $\Pr$ in (A, T) such that $\Pr > \mu H(x)$ is considered a f.s.regular space.

It is clear that every a f.regylar .S is a f.s.regular.s and the converse need not be true see [1].

Remark 2.10

A fuzzy.s.normal.s. need not be a f.s.regular .space. see the example (2.6) which shows that. (A, T) is a f.s.nomal,sp but is not f.s.regular.space.

The following Theorem explain that when (A, T) is a f.s.normal.s. and a f. T1. space then (A, T) will be a f.s.regular.space :-

Theorem 2.11

Given a f.s.normal space (A, T) , its inverse f.s.regular space (A, T) is a f.T1 - space.

Proof

Let (A, T) be a f.s.normal.S. and f.T1 – space.

Let H be a f.c.s in (A, T) and let $\Pr$ be a fuzzy point in (A, T) such that

$\Pr > \mu H(x)$, then $\Pr$ is a f.c.s (properties of f.T1 – space [1])

And since (A, T) is a f.s.normal.space so there exist two f.s.o.s G and F in (A, T) such that $\Pr \leq \mu G(x)$ and $\mu H(x) \leq \mu F(x)$, $\min \{ \mu G(x), \mu F(x) \} = 0$

Thus f.s.regular- space is a (A, T) .

Definition 2.12

(A, T) is said to be f. s*.normal.space if for every two f.s.c.s G and H in (A, T) such that $\min \{ \mu H(x), \mu G(x) \} = 0$, there exist two f.s.o.s V and U in (A, T) such that $\mu H(x) \leq \mu V(x)$ and $\mu G(x) \leq \mu U(x)$, $\min \{ \mu V(x), \mu U(x) \} = 0$

This theorem describes the relationship among f.s.normal.s and f.s*.normal.s:-

Theorem 2.13

Every f.s*.normal.space is a f.s.normal.space.

Proof

Let f.s*.normal.space be (A, T) , let G and H be two f.c.s in (A, T) such that $\min$

$\{\mu H(x), \mu G(x)\} = 0$ then by the remark (1.3.16), G and H are f.s.c.s in (A, T) .

Now, since (A, T) is a f.s*.normal.s so, there exist two f.s.o.s U and V in (A, T) such that $\mu(x) \leq \mu U(x)$ and $\mu H(x) \leq \mu V(x)$, $\min \{\mu U(x), \mu V(x)\} = 0$

Thus (A, T) is a f.s.normal.space

The following theorem shows that when (A, T) is a f.s.normal.space and f.s.space, then (A, T) will be a f.s*.normal space :-

Theorem 2.14

Let (A, T) be a f.s.normal.s, if (A, T) is a f.s.space then (A, T) is a f.s*.normal space.

Proof

f.s.normal.s, f.s.space. be (A, T)

Let G and H be f.s.c.s in (A, T) such that $\min \{\mu H(x), \mu G(x)\} = 0$, then by the proposition (2.7), G and H are f.c.s, $\min \{\mu G(x), \mu H(x)\} = 0$.

In (A, T) , as it is a f.s.normal space, there are two f.s.o.s., U and V , such that $\mu H(x) < \mu V(x)$ and $\mu G(x) \leq \mu U(x)$, and $\min \{\mu U(x), \mu V(x)\} = 0$

therefore (A, T) is a f.s*-normal.s.

Theorem 2.15

Let (A, T) be a f.t.s. if (A, T) is a f.s.space, then (A, T) is a f.normal.space.

Proof

It is clear

Now, we are going to introduce the following lemma which we need it to prove the next theorem: -

Lemma 2.16

Let (A, T) be a f.t.s and let $\mu B(x) \leq \mu A(x)$, then B is a f.sg.o.s if and only if $\mu H(x) \leq \mu B$ whenever $\mu H(x) \leq \mu B(x)$ where H is a f.s.c.s in (A, T) .

Proof

First side

Let B be a f.sg.o.s in (A, T) and let $\mu H \leq \mu B(x)$ where H is a f.s.c.s in (A, T) . we are going to show that $\mu H \leq \mu B$, now since $\mu B(x) \leq \mu H(x)$ the $\mu H(x) \leq \mu B(x)$ since B is a f.sg.c.s and H is a f.s.o.s then $\mu B(x) \leq \mu H(x)$. but $\mu B(x) = \mu(B(x))$ (Remark 1.3.19).

So $\mu(B(x)) \leq \mu H(x)$

And thus, $\mu H(x) \leq \mu B$.

Second side

Let $\mu H(x) \leq \mu B(x)$, for each a f.s.c.s $\mu H(x) \leq \mu B(x)$.

We are going to show that B is a f.sg.o.s., in another meaning, we are going to show

that B is a f.sg.c.s.

Let K be a f.s.o.s in (A, T) such that $\mu B(x) \leq \mu K(x)$, then $\mu K(x) \leq \mu B(x)$.

Since K is a f.s.o.s then K is a f.s.c.s then $\mu K(x) \leq \mu B(x)$, so $\mu B(x) \leq \mu K(x)$ but $\mu(B(x)) = \mu B$ (Remark 1.3.19) then $\mu B(x) \leq \mu K(x)$, so B is a f.sg.c.s therefore B is a f.sg.o.s.

Now we are going to introduce the following theorem

Which gives several properties of f.S*.normal.s by using a f.sg.o.s and f.sg.c.s :-

Theorem 2.17

The following sentences are equivalent if (A, T) is a f.t.s.1) The space (A, T) is f.s*-normal. Two f.sg.o.s, U and V , exist in (A, T) such that $\mu G(x) \leq \mu U(x)$, and $\mu H(x) \leq \mu V(x)$, $\min \{ \mu U(x), \mu V(x) \} = 0$ for every f.s.c.s, G and H in (A, T) such that $\mu H(x), \mu G(x) = 0$.

Proof

In (A, T) , let G and H be f.s.c.s such that $\min \{ \mu G(x), \mu H(x) \} = 0$.

Due to the fact that (A, T) is a f.s*.normal.space, there are two f.s.o.s in (A, T) such that $\mu G(x) < \mu U(x)$ and $\mu H(x) \leq \mu V(x)$ and that $\min \{ \mu U(x), \mu V(x) \} = 0$.

Then, according to proposition (1.5.8), $\min \{ \mu U(x), \mu V(x) \} = 0$, and U and V are f.sg.o.s.

Given f.s.c.s G and $H \in (A, T)$ such that $\{ \mu G(x), \mu H(x) \} = 0$, let them be.

Thus, by (2), two f.sg.o.s, U and V , exist in (A, T)

In such a way that $\min \{ \mu G(x), \mu H(x) \} = 0$ and $\mu H(x), \mu V(x) \leq \mu U(x)$.

Now, let $U_1 = U, V_1 = V$

With V and U being f.s.o.s in (A, T) $\min \{ \mu V_1(x), \mu U_1(x) \} = 0$ and $\mu G(x) \leq \mu U_1(x), \mu H(x) \leq \mu V_1(x)$ according to lemma (2.16), (A, T) is a f.s*.normal.space.

Definition 2.18

A f.t.s (A, T) is said to be fuzzy. s* - regular space (f.s*.regular.s) if for every f.s.c.s H of (A, T) and a fuzzy point Pr in (A, T) , there exist two f.s.o.s U and V in (A, T) such that $\mu H(x) \leq \mu U(x)$ and $Pr > \mu V(x)$, $\min \{ \mu U(x), \mu V(x) \} = 0$.

Remark 2.19

The f.s*.normal space need not be a f.s*.regular.space.

See the example (2.6) where (A, T) is a f.s*.normal space but is not f.s*.regular.s.

Now, we introduce the fuzzy semi – symmetric space which will give us an important condition in the next theorem: -

Definition 2.20

If for each pair of different fuzzy points in a f.t.s (A, T) there exists a fuzzy semi-symmetric space (f.s.sym.s). Pr_1 and Pr_2 if $Pr_1 \leq \mu Pr_2$ then $Pr_2 \leq \mu Pr_1$.

The following lemma gives an equivalent definition to the definition of f.s.sym.s :-

Lemma 2.21 []

Let (A, T) be a f.t.s then the following statement are equivalent:

- 1) (A, T) is a f.s.sym.s.
- 2) every $\Pr$ is a f.sg.c.s in (A, T) .

Remark 2.22

If (A, T) is a f.s*.normal.space and $\min\{\mu H(x), \mu G(x)\} = 0$, then $\min\{\mu H(x), \mu G(x)\} = 0$ there exist two fuzzy s.o.s K and O in (A, T) such that $O(x) > \mu H(x)$ and $\mu K(x) > \mu G(x)$

The following theorem explains that when (A, T) is a f.s*.normal.s and f.s.ym.s, then (A, T) will be a f. s*.regular.s:-

Theorem 2.23

Let (A, T) be a f.s*.normal.s, if (A, T) is a f.s.sym.s then (A, T) is a f.s*.regular.space.

Proof

Let (A, T) be a f.s*.normal.s, f.s.sym.s.

Let B be a f.s.c.s in (A, T) , let $\Pr$ be a fuzzy point in (A, T) such that $\Pr > \mu B(x)$.

So by lemma (2.21) $\Pr$ is a f.sg.c.s in (A, T) .

Now, since B is a f.s.c and $\Pr$ is a f.sg.c.s, $\min\{\Pr, \mu B(x)\} = 0$, so by remark (2.22), there exist two f.s.o.s G and H in (A, T) such that $\mu B(x) \leq \mu G(x)$ and $\Pr \leq \mu H(x)$, $\min\{\mu G(x), \mu H(x)\} = 0$, thus (A, T) is a f.s*.regular space.

Definition 2.24

A f.t.s (A, T) is said to be f.sg.normal.s if for every two f.sg.c.s G and H in (A, T) there exist two f.s.o U and V such that $\mu H(x) \leq \mu V(x)$ and $\mu G(x) \leq \mu U(x)$, $\min\{\mu V(x), \mu U(x)\} = 0$.

The subsequent theorem shows the relationship among the f.sg.normal.s and the two fuzzy spaces f.s.normal.s and f.s*.normal space :-

Theorem 2.25

- 1) Every f. sg.normal.s is a f.s*.normal.s.
- 2) Every f.sg.normal.s is a f.s.normal.s.

Proof

- 1) Let (A, T) be a f.sg. normal.s and let G and H be a f.s.c.s in (A, T) such that $\min\{\mu H(x), \mu G(x)\} = 0$.

By the proposition (1.5.3), G and H are f.sg.c.s in (A, T)

Now, since (A, T) is a f.sg.normal.space there exist two f.s.o.s in (A, T) U and V Such that $\mu H(x) \leq \mu V(x)$ and $\mu G(x) \leq \mu U(x)$, $\min\{\mu V(x), \mu U(x)\} = 0$.

Therefore (A, T) is a f.s*.normal. Space

- 2) The proof is similar to the proof of part (1) and by using the corollary (1.5.4).

Now we introduce the fuzzy semi – $T_{1/2}$ – space (f.s. $T_{1/2}$.space) which will give us an important condition in the next theorem: -

Definition 2.26

A f.t.s (A, T) is said to be fuzzy semi .T₁/2.spce (f.s.T₁/2.s) if every f.sg.c.s in (A, T) is a f.s.c.s

Example 2. 27:

Let $A = \{(a, 0.7), (b, 0.8), (c, 0.9), (d, 0.5)\}$ and let $T = \{\emptyset, A, \{(a, 0.7), (b, 0.), (c, 0.0), (d, 0.0)\}, \{(a, 0.0), (b, 0.8), (c, 0.0), (d, 0.0)\}, \{(a, 0.7), (b, 0.8), (c, 0.0), (d, 0.0)\}, \{(a, 0.7), (b, 0.8), (c, 0.0), (d, 0.5)\}, \{(a, 0.7), (b, 0.8), (c, 0.9), (d, 0.0)\}$
Clearly (A, T) is a f.s.T₁/2. s.

Theorem 2.28

Let (A, T) be a f.s*.normal.s.if (A, T) is a f.s.T₁/2.sp then (A, T) is a f.sg.normal.sp.

Proof:

Assume that (A, T) is a f.s*.normal.s, f.s.T₁/2.sp.

Assume that G and H are f.sg.c.s in (A, T) such that $\min \{\mu H(x), \mu G(x)\} = 0$.

Given that (A, T) is a f.s.T₁/2. s, G and H are f.s.c.s in (A, T) ; hence, $\min \{\mu H(x), \mu G(x)\} = 0$. Additionally, since (A, T) is a f.s*.normal.s., there exist two f.s.o.s in (A, T) such that $\mu G(x) < \mu U(x)$, and $\mu H(x) \leq \mu V(x)$, $\min \{\mu V(x), \mu U(x)\} = 0$.
 (A, T) is a f.sg.normal.s as a result.

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