

ON THE GLOBAL STABILITY OF COMPACT RANDOM SETS VIA UNIFORMLY UNBOUNDED LYAPUNOV FUNCTIONS

HIND ADNAN HASHIM^{1*}, IHSAN JABBAR KADHIM²

Abstract: *The main goal of the work is to examine, via the uniformly unbounded Lyapunov functions, the global stability of compact random sets in random dynamical systems defined on Banach space. In addition, the possibility of a transition of the advantage of the global asymptotic stability of compact random sets between topologically conjugate random dynamical systems is proved. We also showed that compact random sets can be contained in an attraction region. The necessary condition for a compact connected random set to have asymptotic stability is given. The stability of affine random dynamical systems is analyzed utilizing the second Lyapunov approach.*

INTRODUCTION

M. A. Lyapunov, at Kharkov University, presented two approaches to investigate stability in his influential 1892 thesis [17]. The Lyapunov function used in several investigation on the stability of (deterministic) dynamical systems. For example, see Arnold [1], Bacciotti and Rosier [3], Bhatia and Szegö [4], Biles and Schechter [5], Chellaboina, et al. [7], Kloeden [16], Kloeden [15], Oseledets [21], Xu and Yung [23], Parker [22] and Zelik [24]. We will mention in the following some of the literature that dealt with the Lyapunov functions for the purpose of studying the stability of random dynamical systems. Lyapunov's second method was developed for random dynamical systems by Arnold and Schmalfuss [2]. Where they proved that that a necessary and sufficient conditions for the stability of a random set is the existence of the Lyapunov function. Conley [9] discovered that there exists a complete Lyapunov function for a continuous dynamical system with a compact metric space as a phase space that is strictly decreasing on trajectories outside the chain recurrent set and constant on orbits inside the set. For random dynamical systems, Liu [19] created a complete Lyapunov function as long as the underlying separable metric space is equipped with a probability measure. By formulating a complete Lyapunov function, he successfully derived the stochastic variant of Conley's fundamental theorem for dynamical systems.

The Conley index theory is extended to discrete random dynamical systems by Liu [18]. One can explore a number of works that have been given on the topic of studying the stability of random dynamical systems using Lyapunov functions,

¹ Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah, Iraq.
Email: math.phd.22.4@qu.edu.iq

² Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah, Iraq.
ORCID iD: <https://orcid.org/0000-0001-9035-2912?lang=en>

Email: ihsan.kadhimi@qu.edu.iq, Email: ihsankadhimi27@gmail.com

*Correspondence: math.phd.22.4@qu.edu.iq

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including Liu [18], Liu, et al. [20], Crauel, et al. [10], Caraballo, et al. [6], Ju, et al. [13], and Kadhim and Imran [14]. We will structure the rest of the research as follows: In Section 2 of the research, we will present some basic concepts and facts related to the research topic. Section 3 is devoted to the study of the global asymptotic stability of random sets. In Section 4, study the stability of affine random dynamical systems using the Lyapunov function. Throughout, the triple $\Omega \equiv (\Omega, \mathcal{F}, \mathbb{P})$ consider as a probability space and $X \equiv (X, \|\cdot\|)$ be a Banach space. $S(M, \varepsilon) \equiv \{x \in X: \inf_{y \in M(\omega)} \|x - y\| < \varepsilon(\omega)\}$ where $M(\omega)$ is random set in X and ε be a TRV. $S[M, \varepsilon] \equiv \{x \in X: \inf_{y \in M(\omega)} \|x - y\| \leq \varepsilon(\omega)\}$ and $H(M, \varepsilon) \equiv \{x \in X: \inf_{y \in M(\omega)} \|x - y\| = \varepsilon(\omega)\}$.

PRELIMINARIES

As we propose in the subsequent parts, we shall summarize here some basic ideas and general information concerning RDSs. Since RDSs possess attractors, which are generally random sets (see, for example, Crauel, et al. [10], Flandoli and Schmalfuss [12], and Arnold [1]), it is possible to define attractivity and stability for random sets. RDSs are skew-product flows in formal terms, but only in a quantifiable one. Since all topology and continuity are deliberately omitted from the driving or base flow, we can explore other solutions than those proposed by Kloeden [15, 16].

Definition 2.1 [1, 8]: We define the metric dynamical system (MDS) as the measurable action $\theta: \mathbb{R} \times \Omega \rightarrow \Omega$ that is verified $\mathbb{P}(\theta_t B) = \mathbb{P}(B)$ for every $B \in \mathcal{F}$ and $t \in \mathbb{R}$. The MDS is denoted by θ .

Definition 2.2 [1, 8]: A pair (θ, φ) of mappings is called the random dynamical system (RDS) where θ is a MDS and $\varphi: \mathbb{T} \times \Omega \times X \rightarrow X$ is a function verify

$$\varphi(0, \omega) = \text{id}, \varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega), \text{ where } t, s \in \mathbb{R} \text{ and } \omega \in \Omega. \quad (2.1)$$

The function φ called co-cycle over θ .

If, in addition, the functions $\varphi(\cdot, \omega, \cdot): \mathbb{R} \times X \rightarrow X$ is continuous for every $\omega \in \Omega$, then the co-cycle with the MDS θ called random dynamical system (RDS) over θ .

Definition 2.3 [1, 8]: The affine RDS is a pair (θ, φ) such that

$$\varphi(t, \omega)x = \Phi(t, \omega)x + \psi(t, \omega) \quad (2.2)$$

where $\Phi(t, \omega)$ verify (2.1) and $\psi: \mathbb{R} \times \Omega \rightarrow X$ is a measurable mapping. In case of $\psi(t, \omega) \equiv 0$, then (θ, φ) is called linear and it is written by LRDS.

Definition 2.4 [1, 8]: Let (X, d) be a metric space.

(i) if the function $\rho(x, \omega) := \text{dist}_X(x, M(\omega))$ is meaclosed (for every $x \in X$, then the set-valued function $M: \Omega \rightarrow 2^X / \{\emptyset\}$ is called **random set**. When $M(\omega)$ is closed (compact) in X for each $\omega \in \Omega$, then $M(\omega)$ is called a closed (resp. compact) random set.

(ii) A tempered random variable (TRV) is a random variable $\varepsilon: \Omega \rightarrow \mathbb{R}$ with the property that $\lim_{t \rightarrow +\infty} \frac{1}{|t|} \log |\varepsilon(\theta_t \omega)| = 0$.

Definition 2.5 [8] Let $D(\omega)$ be a random set. The random trajectory (or orbit) emanating from $D(\omega)$ is defined by $\gamma_D^t(\omega) := \bigcup_{\tau \geq t} \varphi(\tau, \theta_{-\tau}\omega) D(\theta_{-\tau}\omega)$.

Definiktion 2.6 [8] For a random set $D(\omega)$, we define the (pull-back) omega-limit set of $\gamma_D^t(\omega)$ by the random set.

$$\Gamma_D(\omega) = \left\{ x \in X : \exists \{t_n\} \subset \mathbb{R}^+, \{y_n\} \subset D(\theta_{-t_n}\omega) \ni x = \lim_{n \rightarrow +\infty} \varphi(t_n, \theta_{-t_n}\omega) y_n \right\}.$$

In the following, we will define some types of attractors of random sets similar to what was stated in Arnold and Schmalfuss [2], Crauel and Flandoli [11] and Chueshov [8].

Definition 2.7: A random compact set $M(\omega)$ is called

- (i) **forward invariant** if $\varphi(t, \omega)M(\omega) \subseteq M(\theta_t\omega)$, $t > 0$ and $\omega \in \Omega$
- (i) an **attractor** of the RDS (θ, φ) if there is a TRV δ such that $\lim_{t \rightarrow +\infty} \sup_{y \in S(\theta_{-t}\omega)} \inf_{x \in M(\omega)} \|\varphi(t, \theta_{-t}\omega)y - x\| = 0$, $\omega \in \Omega$, where $S(\omega) := S(M(\omega), \delta(\omega))$,
- (ii) a **uniform attractor** of the RDS (θ, φ) if there exist TRV $\delta(\varepsilon)$ and for each TRV ε , there exists $\tau := \tau(\varepsilon)$ such that

$$\sup_{y \in \gamma_x^t(\omega)} \inf_{x \in M(\omega)} \|\varphi(t, \theta_{-t}\omega)y - x\| < \varepsilon(\omega), \text{ for } x \in S[M, \delta(\theta_{-t}\omega)]$$

- (iii) a **stable** if for any $\varepsilon > 0$ there is a random compact neighborhood K of M such that $\mathbb{P}\{\omega : \sup_{y \in K(\omega)} \inf_{x \in M(\omega)} \|\varphi(t, \theta_{-t}\omega)y - x\| \geq \varepsilon\} < \varepsilon$, and K is forward invariant.
- (iv) an **asymptotically stable (AS)** with respect to (θ, φ) if M is an attractor of (θ, ψ) and positively stable with respect to (θ, φ) .

Definition 2.9. Let M be the compact random set. The set

$$A_M(\omega) := \left\{ x \in X : \lim_{t \rightarrow +\infty} \inf_{y \in M(\omega)} \|\varphi(t_n, \theta_{-t_n}\omega)x - y\| = 0 \right\}$$

is called the region of attraction of M .

Global Stable Compact Random Sets

A non-negative scalar function that is defined on a set's neighborhood and decreases along its trajectories is used to characterize the global stability and asymptotic stability of a given compact random set. This is the fundamental idea behind Lyapunov stability theory. Generally speaking, stability and the different attractor features cannot be described using continuous. However, one can provide extremely strong theorems in the case of asymptotic stability. In this work, we will describe them. Here we will consider the Lyapunov function that is defined in Arnold and Schmalfuss [2] with two additional properties as given below. In the following, we will introduce the definition of the Lyapunov function for the RDS. Indeed, this definition is analogous to the definition of the Lyapunov that is stated in Arnold and Schmalfuss [2], but we will add two additional axioms.

Definition 3.1. Let $A(\omega)$ be an invariant random compact set in the RDS (θ, φ) . An **exponentially Lyapunov function** for $A(\omega)$ is a function $\mathcal{L}: \Omega \times X \rightarrow \mathbb{R}^+$ admits the following axioms:

- (i) $\mathcal{L}(\omega, \cdot): X \rightarrow \mathbb{R}^+$ is continuous for each $\omega \in \Omega$ and $\mathcal{L}(\cdot, x): \Omega \rightarrow \mathbb{R}^+$ is measurable for each $x \in X$.
- (ii) $\lim_{\|x\| \rightarrow \infty} \mathcal{L}(\omega, x) = \infty$ for all ω ($\mathcal{L}$ is uniformly unbounded).
- (iii) $\mathcal{L}(\omega, x) = 0$ for $x \in A(\omega)$, and $\mathcal{L}(\omega, x) > 0$ for $x \notin A(\omega)$ ($\mathcal{L}$ is positive-definite).
- (iv) $\mathcal{L}(\theta_t \omega, \varphi(t, \omega, x)) \leq \mathcal{L}(\omega, x)$ for all $t > 0$ and $x \notin A(\omega)$ ($\mathcal{L}$ is strictly decreasing along trajectory of φ).
- (v) $\mathcal{L}(\theta_t \omega, \varphi(t, \omega, x)) = e^{-t} \mathcal{L}(\omega, x)$ for all $x \in A_M(\omega)$ and $t \in \mathbb{R}$.
- (vi) $\mathcal{L}(\theta_t \omega, \varphi(t, \omega, x)) \leq e^{-\beta t} \mathcal{L}(\omega, x)$ for any $x \in A_M(\omega)$, $t \geq 0$ and $\beta > 0$.

Definition 3.2 Assume that (θ, φ_1) and (θ, φ_2) are two RDSs with phase spaces X and Y respectively. In topological terms, these RDSs are called conjugate if there is a function $T: \Omega \times X \rightarrow Y$ that satisfies:

- (i) $T(\omega, \cdot): X \rightarrow Y$ is a homeomorphism for every $\omega \in \Omega$;
- (ii) $T(\cdot, x): \Omega \rightarrow Y$ and $T^{-1}(\cdot, y): \Omega \rightarrow X$ are measurable for every $x \in X$ and $y \in Y$;
- (iii) $\varphi_2(t, \omega, T(\omega, x)) = T(\theta_t \omega, \varphi_1(t, \omega, x))$ for every $(t, \omega, x) \in \mathbb{R} \times \Omega \times X$. (3.1)

We will compose $(\theta, \varphi_1) \cong_T (\theta, \varphi_2)$.

Lemma 3.3. Let $(\theta, \varphi_1) \cong_T (\theta, \varphi_2)$. If $\mathcal{L}: \Omega \times X \rightarrow \mathbb{R}^+$ is Lyapunov function for a compact random set $A: \Omega \rightarrow X$, then $\mathcal{L}^*: \Omega \times Y \rightarrow \mathbb{R}^+$ defined by $\mathcal{L}^* := \mathcal{L} \circ (Id_\Omega \times T_\omega^{-1})$ is Lyapunov function for a compact random set $T_\omega \circ A: \Omega \rightarrow Y$.

Proof. First note that the set $T_\omega \circ A$ is random and compact because the function T_ω is a homeomorphism.

- (i) The measurability of the function $\mathcal{L} \circ (Id_\Omega \times T_\omega^{-1}): \Omega \times Y \rightarrow \mathbb{R}^+$ follows from the measurability of the functions $\mathcal{L}_x: \Omega \rightarrow \mathbb{R}^+$ and $T_\omega^{-1}: Y \rightarrow X$. With simple calculations it can be proved that $\mathcal{L}^* = \mathcal{L} \circ T_\omega^{-1}$ is also continuous for each $\omega \in \Omega$.
- (ii) $\mathcal{L}^*$ is uniformly unbounded. For, let $\omega \in \Omega$ and $y \in Y$, then there is $x \in X$ with $y = T_\omega(x)$. Note that, if $\|y\| \rightarrow \infty$, then $\|x\| \rightarrow \infty$. So $\lim_{\|y\| \rightarrow \infty} \mathcal{L}^*(\omega, y) = \lim_{\|y\| \rightarrow \infty} \mathcal{L}(\omega, T_\omega^{-1}(y)) = \lim_{\|x\| \rightarrow \infty} \mathcal{L}(\omega, x) = \infty$ for all ω .
- (iii) $\mathcal{L}^*$ is positive-definite. For, if $y \in T_\omega A(\omega)$, then $\mathcal{L}^*(\omega, y) = \mathcal{L}(\omega, T_\omega^{-1}(y)) = \mathcal{L}(\omega, x) = 0$ for some $x \in A(\omega)$. Now suppose that $y \notin T_\omega A(\omega)$, then $\mathcal{L}(\omega, T_\omega^{-1}(y)) > 0$. This implies that $\mathcal{L}^*(\omega, y) > 0$ for every $y \notin T_\omega A(\omega)$.
- (iv) For, let $y \notin T_\omega A(\omega)$, then $T_\omega^{-1}(y) \notin A(\omega)$. By simple calculations we find that $\mathcal{L}^*(\theta_t \omega, \varphi_2(t, \omega, y)) \leq \mathcal{L}^*(\omega, y)$.

Thus (iv) of Definition (3.1) holds.

- (v) Let for all $y \in T_\omega A_M(\omega)$ and $t \in \mathbb{R}$. Then there is $x \in A_M(\omega)$ such that $y = T_\omega x$

$$\mathcal{L}^*(\theta_t \omega, \varphi_2(t, \omega, y)) = \mathcal{L} \circ (Id_\Omega \times T_\omega^{-1})(\theta_t \omega, \varphi_2(t, \omega, y)) = \mathcal{L}(\theta_t \omega, T_\omega^{-1} \varphi_2(t, \omega, y)) = \mathcal{L}(\theta_t \omega, \varphi_1(t, \omega, T_\omega^{-1} y)) \text{ (by (3.1))}$$

$= e^t \mathcal{L}(\omega, T_\omega^{-1} y) = e^t \mathcal{L}^*(\omega, y)$ (vi) Let $y \in T_\omega A_M(\omega)$, $t \geq 0$ and $\beta > 0$. Then there is

$x \in A_M(\omega)$ such that $y = T_\omega x$

$$\mathcal{L}^*(\theta_t \omega, \varphi_2(t, \omega, y)) = \mathcal{L} \circ (Id_\Omega \times T_\omega^{-1})(\theta_t \omega, \varphi_2(t, \omega, y)) = \mathcal{L}(\theta_t \omega, T_\omega^{-1} \varphi_2(t, \omega, y)) = \mathcal{L}(\theta_t \omega, \varphi_1(t, \omega, T_\omega^{-1} y)) \text{ (by (3.1))}$$

$$\leq e^{-\beta t} \mathcal{L}(\omega, T_\omega^{-1} y) = e^{-\beta t} \mathcal{L}^*(\omega, y)$$

This completes the proof.

Lemma 3.4 Let (θ, φ) be an RDS and $K(\omega)$ be any random set. Let $\mathcal{L}: \Omega \times K \rightarrow \mathbb{R}$ be any continuous function defined on K such that $\mathcal{L}(\theta_t \omega, \varphi(t, \omega)x) < \mathcal{L}(\omega, x)$ for all $t > 0$ and $x \notin K(\omega)$. Whenever $\gamma_x^{[0, t]}(\omega) \subset K(\omega)$, $t \geq 0$. Then if some x , with $\overline{\gamma_x^{\mathbb{R}^+}(\omega)} \subset K(\omega)$, we have $\mathcal{L}(\omega, y) = \mathcal{L}(\omega, z)$ for every $y, z \in \Gamma_x^+(\omega)$.

Proof. Assume that $\mathcal{L}(\omega, y) < \mathcal{L}(\omega, z)$. There are indeed sequence $\{t_n\}$ and $\{\tau_n\}$ in $\mathbb{R}$ such that $t_n \rightarrow +\infty$, $\tau_n \rightarrow +\infty$ and $\varphi(t_n, \omega)x \rightarrow y$, $\varphi(\tau_n, \omega)x \rightarrow z$. We may assume by taking a subsequence that $t_n < \tau_n$ for each n . Then clearly $\mathcal{L}(\theta_{t_n} \omega, \varphi(t_n, \omega)x) \geq \mathcal{L}(\theta_{\tau_n} \omega, \varphi(\tau_n, \omega)x)$ as follows:

$$\varphi(\tau_n, \omega)x = \varphi((\tau_n - t_n), \theta_{t_n} \omega) \circ \varphi(t_n, \omega)x,$$

$\tau_n - t_n > 0$, and

$\{\varphi(s, \theta_{t_n} \omega, \varphi(t_n, \omega)x) : s \in [0, \tau_n - t_n]\} \subset K(\omega)$, Thus proceeding to the limit, we have continuity of $\mathcal{L}$, $\mathcal{L}(\omega, y) \geq \mathcal{L}(\omega, z)$. This contradicts the original assumption and the limit is proved.

Theorem 3.5 If $M: \Omega \rightarrow 2^X$ is a compact AS invariant random set, then there is $\mathcal{L}: \Omega \times A_M(\omega) \rightarrow \mathbb{R}^+$ satisfying conditions (i)-(v) of Definition (3.1)

Proof. Consider any function $\mathcal{L}: \Omega \times N \rightarrow \mathbb{R}^+$, where N be any nhd of M and satisfying (i)-(v) of Definition (3.1) and $\{\varphi(s, \omega, x) : s \in [0, t]\} \subset N(\omega)$. Let ε be a TRV chosen so that $S[M(\omega), \varepsilon(\omega)]$ is a compact random subset of $Int(N(\omega))$. Set $m(\varepsilon) := \min\{\mathcal{L}(\omega, x) : x \in H(M(\omega), \varepsilon(\omega))\}$. Clearly $m(\varepsilon) > 0$. Now let $0 < \alpha < m(\varepsilon)$, and set

$$P_\alpha(\omega) = \{x \in N(\omega) : \mathcal{L}(\omega, x) \leq \alpha\} \cap S[M(\omega), \varepsilon(\omega)].$$

Consider the random set $P_\alpha(\omega)$. Note that $\partial P_\alpha(\omega) = \{x \in X : \inf_{z \in M(\theta_t \omega)} \|x - z\| \leq \varepsilon(\omega) \text{ and } \mathcal{L}(\omega, x) = \alpha\}$. We assert that if $x \in A_M(\omega) - M(\omega)$ there exists a unique $\tau(x) \in \mathbb{R}$ such that $\varphi(\tau(x), \theta_{-\tau(x)} \omega)x \in \partial P_\alpha(\omega)$. This yields from the fact that if $\varphi(\tau(x), \theta_{-\tau(x)} \omega)x \in \partial P_\alpha(\omega)$, then

$$\mathcal{L}(\theta_{\tau(x)+t} \omega, \varphi(\tau(x) + t, \omega)x) < \mathcal{L}(\theta_{-\tau(x)} \omega, \varphi(\tau(x), \omega)x) = \alpha \text{ for } t > 0.$$

As $P_\alpha(\omega)$ is forward invariant and $\mathcal{L}(\omega, y) = \alpha$ for $y \in \partial P_\alpha(\omega)$, we conclude that $\varphi(\tau(x) + t, \omega, x) \in Int(P_\alpha(\omega))$, for all $t > 0$. This creates uniqueness of $\tau(x)$ for every x in the domain of τ .

Since $P_\alpha(\omega)$ is a random nhd of $M(\omega)$, then τ is defined on $A_M(\omega) - P_\alpha(\omega)$, thus

$\gamma_x^{\mathbb{R}}(\omega) \subset A_M(\theta_t\omega)$ with $x \notin P_\alpha(\omega)$, must intersect $\partial P_\alpha(\omega)$. In fact τ is well-defined and finished on $P_\alpha(\omega)$. For $x \in \text{int}P_\alpha(\omega) - M(\omega)$, if there is no $\tau(x)$ with $\varphi(\tau(x), \theta_{-\tau(x)}\omega)x \in P_\alpha(\omega)$, then $\gamma_x^{\mathbb{R}}(\omega) \subset \text{int}P_\alpha(\theta_t\omega) - M(\theta_t\omega) \subset P_\alpha(\omega)$. $P_\alpha(\omega)$ being compact, $\Gamma_x^-(\omega) \neq \emptyset$, $\Gamma_x^-(\omega) \subset P_\alpha(\omega)$, but $\Gamma_x^-(\omega) \cap M(\theta_t\omega) = \emptyset$ (otherwise $M(\theta_t\omega)$ will be unstable). Since $\Gamma_x^-(\omega)$ is compact invariant random set, so that if $y \in \Gamma_x^-(\omega)$ then $\Gamma_y^+(\omega) \neq \emptyset$ and $\Gamma_y^+(\omega) \subset \Gamma_x^-(\omega)$. Then on one hand we have $\Gamma_y^+(\omega) \cap M(\theta_t\omega) = \emptyset$ and on the other hand $\Gamma_y^+(\omega) \subset M(\theta_t\omega)$ as $y \in A_M(\theta_t\omega)$. This contradicts $\Gamma_y^+(\omega) \neq \emptyset$. Thus $\gamma_x^{\mathbb{R}}(\omega) \cap \partial P_\alpha(\omega) \neq \emptyset$ and $\tau(x)$ is defined for each $x \in A_M(\omega) - M(\omega)$. Note that $A_M(\omega) - M(\omega)$ is invariant, as $M(\omega)$ and $A_M(\omega)$ are both invariant. For $x \in A_M(\omega) - M(\omega)$ and $\tau \in \mathbb{R}$ observe now that $\tau(\varphi(t, \omega)x) = \tau(x) - t$. This is because any trajectory $\gamma_x^{\mathbb{R}}(\omega)$ in $A_M(\theta_t\omega) - M(\theta_t\omega)$ intersects $\partial P_\alpha(\omega)$ at exactly one point. Thus

$$(\varphi(t, \omega, x))(\tau(\varphi(t, \omega)x) = \varphi(\tau(x), \theta_{-\tau(x)}\omega)x$$

that is, by the homeomorphism axiom

$$\varphi(t + \tau(\varphi(t, \omega, x)), \omega)x = \varphi(\tau(x), \theta_{-\tau(x)}\omega)x.$$

As $\gamma_x^{\mathbb{R}}(\omega)$ cannot be a random fixed point or periodic, we have $t + \tau(\varphi(t, \omega, x)) = \tau(x)$. This shows further that $\tau(\varphi(t, \omega, x))$ is a continuous function of t and $\tau(\varphi(t, \omega)x) \rightarrow \pm\infty$ as $t \rightarrow \pm\infty$. We assert that on $A_M(\theta_t\omega) - M(\theta_t\omega)$, $\tau(x)$ is continuous. For any $x \in A_M(\theta_t\omega) - M(\theta_t\omega)$, and $\varepsilon > 0$, the point $y \equiv \varphi(\tau(x) + \varepsilon, \theta_{-\tau(x)-\varepsilon}\omega)x \in \text{int}(P_\alpha(\omega))$. There is therefore an nhd $N_y(\omega)$ of y such that $N_y(\omega) \subset P_\alpha(\omega)$. Then $N^+(\omega) = N_y(-\tau(x) - \varepsilon)$ is an nhd of x and for every $z \in N^+(\omega)$, $\tau(z) \leq \tau(x) + \varepsilon$. Again $z \equiv \varphi(\tau(x) - \varepsilon, \theta_{-\tau(x)+\varepsilon}\omega)x \in A_M(\theta_t\omega) - P_\alpha(\omega)$, the latter set is open. Therefore there exists an nhd $N_z(\omega)$ of z so that $N_z(\omega) \subset (A_M(\theta_t\omega) - P_\alpha(\omega))$. Then $N^- = N_z(-\tau(x) + \varepsilon)$ is a nhd of x and for each $z \in N^-$ we have $\tau(z) \geq \tau(x) - \varepsilon$. Thus if z is in the neighborhood $N_\varepsilon = N^- \cap N^+$ of x we have $\tau(x) - \varepsilon \leq \tau(z) \leq \tau(x) + \varepsilon$. Consequently τ is continuous on $A_M(\theta_t\omega) - M(\theta_t\omega)$. We now show that $\tau(x) \rightarrow -\infty$ as $x \rightarrow M(\theta_t(\omega))$, $x \in A_M(\theta_t\omega) - M(\theta_t\omega)$. If this is false, then there is a $T > 0$ and $\{x_n\}$ in $A_M(\theta_t\omega) - M(\theta_t\omega)$, such that $x_n \rightarrow x \in M(\omega)$ and $v - T \leq \tau(x_n) \leq 0$. Since the sequence $\{\tau(x_n)\}$ is bounded, then it admits a convergent subsequence. Suppose that $\tau(x_n) \rightarrow \tau$, where $-T \leq \tau \leq 0$. Since $\varphi(\cdot, \omega, \cdot): \mathbb{R} \times X \rightarrow X$ is continuous for every ω , then $\varphi(\tau(x_n), \omega, x_n) \rightarrow \varphi(t, \omega, x)$. As $M(\theta_t\omega)$ is invariant $\varphi(t, \omega, x) \in M(\theta_t\omega)$, on the other hand $\varphi(\tau(x_n), \omega, x_n) \in \partial P_\alpha(\omega)$ which is compact. Therefore $\varphi(t, \omega, x) \in \partial P_\alpha(\omega)$ but $\partial P_\alpha(\omega) \cap M(\theta_t\omega) = \emptyset$ a contradiction. Define $\mathcal{L}: \Omega \times A_M(\omega) \rightarrow \mathbb{R}^+$ as follows

$$\mathcal{L}(\omega, x) := \begin{cases} 0, & x \in M(\omega) \\ e^{\tau(x)}, & x \in A_M(\omega) - M(\omega) \end{cases}.$$

Then $\mathcal{L}$ is continuous on $A_M(\omega)$. Also $\mathcal{L}(\omega, x) > 0$ whenever $x \notin M(\omega)$ for every ω , and

$$\mathcal{L}(\theta_t\omega, \varphi(t, \omega, x)) = e^{\tau(\varphi(t, \omega, x))} = e^{\tau(x)-t} = e^{-t}\mathcal{L}(\omega, x).$$

To show that $\mathcal{L}(\omega, x)$ is uniformly unbounded, remember that

$$A_M(\omega) = \gamma_{P_\alpha}^{[-n,0]}(\omega), n = 1, 2, 3, \dots$$

Each $\gamma_{P_\alpha}^{[-n,0]}(\omega)$ is forward invariant and compact random set. If $x \notin \gamma_{P_\alpha}^{[-n,0]}(\omega)$, then $\tau(x) > n$, hence $\mathcal{L}(\omega, x) > e^n$. ■

As a result of the foregoing results, we can establish the following two theorems for GAS. It is worth remarking that, up to now, we will consider M as a compact invariant random set.

Theorem 3.6. A random set $M(\omega)$ in X is GAS if and only if there is $\mathcal{L}: \Omega \times X \rightarrow \mathbb{R}^+$ having the properties (i)-(v) of Definition (3.1)

Proof. The sufficiency. If $\mathcal{L}(\omega, x)$ is uniformly unbounded on $N(\omega)$, then for any $\alpha > 0$, the set $K_\alpha(\omega) = \{x: \mathcal{L}(\omega, x) \leq \alpha\} = \mathcal{L}_\omega^{-1}([0, \alpha])$ is compact. Then $K_\alpha(\omega) \subset A_M(\omega)$. Now $N(\omega) = \bigcup_{n=1}^\infty K_n(\omega)$, and since each $K_n(\omega) \subset A(M)$ we have $N(\omega) \subset A_M(\omega)$. Lastly if $N(\omega)$ is invariant we must have $N(\omega) = A_M(\omega)$ as $N(\omega)$ is a nhd of $M(\omega)$. To see that $M(\omega)$ is an attractor, choose any compact positively invariant nhd $K(\omega)$ of $M(\omega)$ with $K(\omega) \subset N(\omega)$. Then for any $x \in K(\omega)$, $\emptyset \neq \Gamma_x^+(\omega) \subset K(\omega)$, and Lemma 3.4 shows that $\mathcal{L}(\omega, x)$ is constant on $\Gamma_x^+(\omega) \subset M(\omega)$. Thus $M(\omega)$ is attractor and consequently GAS.

The necessity follows from Theorem 3.5. As a final result on asymptotic stability we wish to prove the following theorem whose proof illustrates yet another technique of proving the existence of special kinds of Lyapunov functions.

Theorem 3.7. If $M(\omega) \subset X$ is an AS random set with the region of attraction $A_M(\omega)$, then for every $\beta > 0$ there is a function $\mathcal{L}: \Omega \times A_M(\omega) \rightarrow \mathbb{R}^+$ on having the properties (i)-(iv) and (vi) of Definition (3.1).

Proof. Let P be a forward invariant compact random nhd of $M(\omega)$, $P(\omega) \subset A_M(\omega)$. Then there is an integer n_0 such that $S[M(\omega), \frac{1}{n}] \subset P(\omega)$ for $n \geq n_0$. For each $n \geq n_0$ define $\psi_n: \Omega \times P(\omega) \rightarrow \mathbb{R}^+$ by

$$\psi_n(\omega, x) := \sup_{t \geq 0} \inf_{y \in \bar{S}_n(\theta_{-t}\omega)} \|\varphi(t, \theta_{-t}\omega)x - y\| e^{\beta t}$$

where $\bar{S}_n(\omega) := \{y \in X: \inf_{z \in M(\omega)} \|y - z\| \leq 1/n\}$. To show that ψ_n is well defined. Indeed, for every $n \geq n_0$, there exists a $T_n > 0$ with $\inf_{z \in M(\omega)} \|\varphi(t, \theta_{-t}\omega)x - z\| \leq 1/n$ for every $x \in P(\omega)$ and $t > T_n$. For otherwise for some n^* there will be a sequence $\{x_n\}$ in $P(\omega)$ and a sequence $t_n \rightarrow +\infty$, such that $\varphi(t_n, \theta_{-t_n}\omega)x_n \in P(\omega) - \bar{S}_{n^*}(\omega)$. We may assume that $x_n \rightarrow x \in P(\omega)$, and $\varphi(t_n, \theta_{-t_n}\omega)x_n \rightarrow y \in P - S_{n^*}(\omega)$. Then $y \in J_x^+(\omega)$, $x \in A_M(\omega)$, but $y \notin M(\omega)$. This contradicts the fact that $M(\omega)$ is a uniform attraction. Hence indeed for each $n \geq n_0$

$$\psi_n(\omega, x) := \sup_{T_n \geq t \geq 0} \inf_{y \in \bar{S}_n(\theta_{-t}\omega)} \|\varphi(t, \theta_{-t}\omega)x - y\| e^{\beta t},$$

and $\psi_n(\omega, x)$ is well-defined. It is now clear to see that $\psi_n(\omega, \cdot)$ is continuous for every ω and $\psi_n(\cdot, x)$ and furthermore $\psi_n(\omega, x)$ satisfies the inequality $\psi_n(\theta_t\omega, \varphi(t, \omega)x) \leq e^{-\beta t} \psi_n(\omega, x)$, for all $(\omega, x) \in \Omega \times P(\omega)$, $t \geq 0$.

$$\begin{aligned} \psi_n(\theta_t\omega, \varphi(t, \omega)x) &:= \sup_{\tau \geq 0} \inf_{y \in \bar{S}_n(\theta_{-t}\omega)} \|\varphi(\tau, \theta_{-t}\theta_t\omega) \circ \varphi(t, \omega)x - y\| e^{\beta t} = \\ &\sup_{\tau \geq 0} \inf_{y \in \bar{S}_n(\theta_{-t}\omega)} \|\varphi(\tau, \theta_{-t+t}\omega) \circ \varphi(t, \omega)x - y\| e^{\beta t} = \\ \sup_{\tau \geq t} \inf_{y \in \bar{S}_n(\theta_{-t}\omega)} \|\varphi(\tau, \theta_{-t}\omega)x - y\| e^{\beta(\tau-t)} &= e^{-\beta t} \sup_{\tau \geq t} \inf_{y \in \bar{S}_n(\theta_{-t}\omega)} \|\varphi(\tau, \theta_{-t}\omega)x - y\| e^{\beta \tau} \leq e^{-\beta t} \psi_n(\omega, x) \end{aligned}$$

for all $(\omega, x) \in \Omega \times P(\omega)$, $t \geq 0$.

Finally let $\max\{\psi_n(\omega, x): (\omega, x) \in \Omega \times P(\omega)\} = M_n$. Define $\psi: \Omega \times P(\omega) \rightarrow \mathbb{R}^+$ by

$$\psi(\omega, x) := \sum_{n=n_0}^{\infty} \frac{1}{2^n} \frac{\psi_n(\omega, x)}{M_n}$$

This $\psi(\omega, \cdot)$ is continuous on $P(\omega)$ for every ω , $\psi(\cdot, x)$ is measurable on Ω for every x and satisfies $\psi(\theta_t \omega, \varphi(t, \omega)x) \leq e^{-\beta t} \psi(\omega, x)$ for all $(\omega, x) \in \Omega \times P(\omega)$, $t \geq 0$. Moreover, $\psi(\omega, x) = 0$ whenever $x \in M(\omega)$, $\psi(\omega, x) > 0$ whenever $x \in P(\omega) - M(\omega)$. Now choose any $\alpha > 0$ sufficiently small such that

$$P_\alpha(\omega) := \{x \in P(\omega): \psi(\omega, x) \leq \alpha\} = P(\omega) \cap \psi_\omega^{-1}([0, \alpha])$$

is in $\text{Int } P(\omega)$. Define $\mathcal{L}: \Omega \times A_M(\omega) \rightarrow \mathbb{R}^+$ by

$$\mathcal{L}(\omega, x) := \begin{cases} \psi(\omega, x), & \text{if } x \in P_\alpha(\omega), \\ \alpha e^{\beta \tau(x)}, & \text{if } x \notin P_\alpha(\omega), \end{cases}$$

where $\tau(x)$ is the unique number with $\varphi(\tau(x), \omega)x \in \partial P_\alpha(\omega)$. It is simple to verify that this $\mathcal{L}(\omega, x)$ possesses all the qualities needed by the theorem.

Theorems that enable the identification of $\partial A_M(\omega)$ in the event of an AS compact, connected random set $M(\omega)$ in X will be discussed next.

Theorem 3.8. Consider an RDS (θ, φ) . Let $M(\omega) \subset X$ be a connected random set. Then a necessary condition for $M(\omega)$ to be AS and the open invariant set $A_M(\omega)$ its region of attraction is the existence of two functions $\mathcal{L}, \mathcal{M}: \Omega \times X \rightarrow \mathbb{R}$ which have the following properties:

$$\mathcal{L}_\omega(x) \text{ is continuous on } A_M(\omega) \text{ for every } \omega. \quad (3.1)$$

$$\mathcal{M}_\omega(x) \text{ is continuous on } X \text{ for every } \omega. \quad (3.2)$$

$$-1 < \mathcal{L}(\omega, x) < 0 \text{ if } x \in A_M(\theta_t \omega) - M(\theta_t \omega). \quad (3.3)$$

$$\mathcal{M}(\omega, x) > 0 \text{ if } x \in X - M(\theta_t \omega). \quad (3.4)$$

$$\mathcal{L}(\omega, x) = \mathcal{M}(\omega, x) = 0 \text{ if } x \in M(\theta_t \omega). \quad (3.5)$$

$\mathcal{L}(\omega, x)$ is strictly increasing as $\inf_{z \in M(\theta_t \omega)} \|x - z\|$ increases, that is. given any $\sigma_1 > 0$, there is $\sigma_2 > 0$ with $\mathcal{L}(\omega, x) < \sigma_2$ for $\inf_{z \in M(\theta_t \omega)} \|x - z\| > \sigma_1$. (3.6)

For each sequence $\{x_n\} \subset A_M(\theta_t \omega)$ such that either $x_n \rightarrow x \in \partial A_M(\theta_t \omega)$ or $\lim_{n \rightarrow +\infty} \inf_{z \in M(\theta_t \omega)} \|x_n - z\| = +\infty$, $\lim_{n \rightarrow +\infty} \mathcal{L}(\omega, x_n) = -1$. (3.7)

$$\left(\frac{d\mathcal{L}(\theta_t \omega, \varphi(t, \omega, x))}{dt} \right)_{t=0} = \lim_{t \rightarrow 0} \frac{\mathcal{L}(\theta_t \omega, \varphi(t, \omega, x)) - \mathcal{L}(\omega, x)}{t} = \mathcal{M}(x, \omega)(1 + \mathcal{L}(\omega, x)). \quad (3.8)$$

Proof. Let δ_1 be a TRV such that $S[M(\omega), \delta_1(\omega)] \subset A_M(\theta_t \omega)$. Assume

$$\lambda(t) = \sup \{ \inf_{z \in M(\theta_t \omega)} \|\varphi(t, \omega, x) - z\| : x \in S[M(\theta_t \omega), \delta_1(\omega)] - M(\theta_t \omega) \}, \quad (3.9)$$

for each $t \geq 0$ such $\lambda(t)$ exists and is positive. Since $M(\omega)$ is AS and $S[M(\theta_t\omega), \delta_1(\omega)] \subset A_M(\theta_t\omega)$, $\lambda(t) \leq \varepsilon_1$ for $0 \leq t \leq +\infty$, and $\lim_{t \rightarrow +\infty} \lambda(t) = 0$. Then for each integer k one can find $t_k > 0$ such that

$$\lambda(t) < \varepsilon_1/2^k \text{ for all } t \text{ with } t \geq t_k. \quad (3.10)$$

In addition, we can find an increasing subsequence $\{t_k\}$ such that $t_k \rightarrow +\infty$. Assume next

$$l_k(t) = [3\varepsilon_1 t_{k+1} - \varepsilon_1(t_k + t)] \setminus 2^k(t_{k+1} - t_k), \quad (3.11)$$

for $t_k \leq t \leq t_{k+1}$ and $k \geq 1$. The function $l_k(t)$ is increasing in the interval of definition. In addition, since

$$l_k(t) \geq [3\varepsilon_1 t_{k+1} - \varepsilon_1(t_k + t)] \setminus 2^k(t_{k+1} - t_k) = \frac{\varepsilon_1}{2^{k+1}} > \lambda(t), \quad (3.12)$$

in the interval of definition, we have

$$\lambda(t) < l_k(t) \text{ for } t_k \leq t \leq t_{k+1} \text{ and } k \geq 1. \quad (3.13)$$

For $k = 1$ we can assume that the function $l_1(t)$ is defined on the interval $(-\infty, t_2)$. Since $l_1(t) > \varepsilon_1 \geq \lambda(t)$, the function

$$L(t) := \begin{cases} l_k(t), & t_k \leq t \leq t_{k+1}, k \geq 1 \\ l_1(t), & -\infty < t \leq t_2 \end{cases} \quad (3.14)$$

for each $t > 0$ satisfies the inequality

$$\lambda(t) < L(t). \quad (3.15)$$

Such function $L(t)$ is continuous on $\mathbb{R}$ and in addition it is strictly decreasing from $+\infty$ to 0. From the equation (3.9) and the inequality (3.15) it then follows that

$$\inf_{z \in M(\theta_t\omega)} \|\varphi(t, \omega, x) - z\| < L(t) \quad (3.16)$$

for $t \in \mathbb{R}^+$ and $x \in S(M(\theta_t\omega), \delta_1(\omega))$. Let $L^{-1}(s)$ be the inverse function of $L(t)$. Then $L^{-1}(s)$ is continuous and strictly decreasing on the interval $0 < s < +\infty$. Consider the function

$$\xi(s) = s e^{-L^{-1}(s)}, \quad 0 < s < +\infty, \quad (3.17)$$

which is positive, continuous, strictly increasing and such that $\xi(s) \rightarrow \infty$ as $s \rightarrow 0$. We shall next assume $\xi(0) = 0$ and therefore define $\xi(s)$ on the whole interval $0 < s < +\infty$. Consider the function

$$\mathcal{M}(\omega, x) = \xi(\inf_{z \in M(\theta_t\omega)} \|x - z\|) = \inf_{z \in M(\theta_t\omega)} \|x - z\| \exp\{-L^{-1}(\inf_{z \in M(\theta_t\omega)} \|x - z\|)\}. \quad (3.18)$$

Then $\mathcal{M}$ fulfills all situations of the theorem since it is defined and continuous for all $x \in X$, it is positive for all $x \in X - M(\theta_{-t}\omega)$, it vanishes for $x \in M(\theta_{-t}\omega)$ and it is strictly increasing as $\inf_{z \in M(\theta_t\omega)} \|x - z\|$ increases. Consider next the function

$$\mathcal{L}(\omega, x) = \exp \left[- \int_0^{+\infty} \xi(\inf_{z \in M(\theta_t\omega)} \|\varphi(t, \omega, x) - z\|) dt \right] - 1, \quad (3.19)$$

we will prove that, if for each $x \in M(\theta_t\omega)$, $\mathcal{L}(\omega, x)$ exists, we have

$$1 + \mathcal{L}(\theta_t, \varphi(t, \omega, x)) = [1 + \mathcal{L}(\omega, x)] \exp \left[\int_0^t \xi(d(\varphi(\tau, \omega, x), M(\theta_\tau(\omega))) d\tau \right]. \quad (3.20)$$

From (3.19) it follows

$$\begin{aligned} 1 + \mathcal{L}(\omega, x) &= \exp \left[- \int_0^t \xi(\inf_{z \in M(\theta_\tau\omega)} \|\varphi(\tau, \omega, x) - z\|) d\tau \right] \times \\ &\quad \exp \left[- \int_t^{+\infty} \xi(\inf_{z \in M(\theta_\tau\omega)} \|\varphi(\tau, \omega, x) - z\|) d\tau \right]. \end{aligned} \quad (3.21)$$

Hence, if we let $\tau = t + s$, we have

$$\begin{aligned} (1 + \mathcal{L}(\omega, x)) \exp \left[\int_0^t \xi(\inf_{z \in M(\theta_\tau\omega)} \|\varphi(\tau, \omega, x) - z\|) d\tau \right] &= \\ \exp \left[- \int_0^{+\infty} \xi(\inf_{z \in M(\theta_\tau\omega)} \|\varphi(t + s, \omega, x) - z\|) ds \right] &= \\ \exp \left[- \int_0^{+\infty} \xi(\inf_{z \in M(\theta_\tau\omega)} \|\varphi(t, \theta(s)\omega), \varphi(s, \omega, x) - z\|) ds \right] &= 1 + \\ \mathcal{L}(\theta_t\omega, \varphi(t, \omega, x)) &\quad (3.22) \end{aligned}$$

where the equality $\varphi(t + s, \omega, x) = \varphi(t, \theta(s)\omega), \varphi(s, \omega, x)$ has been used. In order for (3.20) to be true it is then enough that for each $x \in A_M(\omega)$, $\mathcal{L}(\omega, x)$ exists, i.e. that the integral $\int_0^{+\infty} \xi(\inf_{z \in M(\theta_t\omega)} \|\varphi(t, \omega)x - z\|) dt$ exists for each $x \in A_M(\theta_t\omega)$. For each $x \in A_M(\theta_t\omega)$ it is possible to find $\tau(x) > 0$ such that $\varphi(\tau(x), \omega, x) \in S[M(\theta_{-\tau(x)}\omega), \delta_1(\omega)]$, for $t \geq \tau(x)$. It follows that

$$\begin{aligned} \int_0^{+\infty} \xi(\inf_{z \in M(\theta_t\omega)} \|\varphi(t, \omega)x - z\|) dt &= \int_0^{\tau(x)} \xi(\inf_{z \in M(\theta_t\omega)} \|\varphi(t, \omega)x - z\|) dt \\ &\quad + \int_{\tau(x)}^{+\infty} \xi(\inf_{z \in M(\theta_t\omega)} \|\varphi(t, \omega)x - z\|) dt. \end{aligned} \quad (3.23)$$

If we change the integration variable in the second integral from t into $s + \tau(x)$ and notice that, as usual,

$$\varphi(s + \tau(x), \omega, x) = \varphi(\tau(x), \theta(s)\omega), \varphi(s, \omega, x),$$

we have that

$$\begin{aligned} \int_{\tau(x)}^{+\infty} \xi(\inf_{z \in M(\theta_t\omega)} \|\varphi(t, \omega)x - z\|) dt &= \int_0^{+\infty} \xi(\inf_{z \in M(\theta_s\omega)} \|\varphi(s + \tau(x), \omega)x - \\ z\|) ds &= \int_0^{+\infty} \xi(\inf_{z \in M(\theta_s\omega)} \|\varphi(\tau(x), \theta_s\omega) \circ \varphi(s, \omega)x - z\|) ds \end{aligned} \quad (3.24)$$

Now $\varphi(\tau(x), \omega)x \in S(M(\theta_{\tau(x)}\omega), \delta_1(\omega))$ and hence from (3.16) we have

$$L(s) > \inf_{z \in M(\theta_s\omega)} \|\varphi(\tau(x), \theta_s\omega) \circ \varphi(s, \omega)x - z\| \text{ for } s \in \mathbb{R}^+. \quad (3.25)$$

Now, since L^{-1} is strictly decreasing, from the inequality (3.25) so

$$s = L^{-1}(L(s)) < L^{-1}\{\inf_{z \in M(\theta_t \omega)} \|\varphi(\tau(x), \theta_s \omega) \circ \varphi(s, \omega)x - z\|\}, \quad (3.26)$$

from which we have that

$$\begin{aligned} \inf_{z \in M(\theta_t \omega)} \|\varphi(\tau(x), \theta_s \omega) \circ \varphi(s, \omega)x - z\| e^{-\varepsilon} &> \inf_{z \in M(\theta_t \omega)} \|\varphi(\tau(x), \theta_s \omega) \circ \\ \varphi(s, \omega)x - z\| \times \exp [-L^{-1}\{\inf_{z \in M(\theta_t \omega)} \|\varphi(\tau(x), \theta_s \omega) \circ \varphi(s, \omega)x - z\|\}] &= \\ \xi(\inf_{z \in M(\theta_t \omega)} \|\varphi(\tau(x), \theta_s \omega) \circ \varphi(s, \omega)x - z\|). \end{aligned} \quad (3.27)$$

Thus, since $L(t) < \delta$ for $t > 0$, because of the inequalities (3.16) and (3.27), hence the integral (3.24) is less than δ and therefore converges. It follows that for each $x \in A_M(\omega)$, $\int_0^{+\infty} \xi(\inf_{z \in M(\theta_t \omega)} \|\varphi(t, \omega)x - z\|) dt$ exists, hence for each $x \in A_M(\omega)$, $\mathcal{L}(\omega, x)$ exists, and the equality is proved.

We want to show next that $\mathcal{L}(\omega, x)$ is continuous for each $x \in A_M(\omega)$, For that it is enough to prove that

$$\mathcal{L}^*(\omega, x) = \log(1 + \mathcal{L}(\omega, x)) = - \int_0^{+\infty} \xi(\inf_{z \in M(\theta_t \omega)} \|\varphi(t, \omega)x - z\|) dt \quad (3.28)$$

is continuous for every $x \in A_M(\omega)$, that is, for each $x \in A_M(\omega)$ and $\varepsilon > 0$ there is $\delta_1 > 0$ so that, if $\|y - x\| < \delta_1$, then

$$\begin{aligned} |\mathcal{L}^*(\omega, y) - \mathcal{L}^*(\omega, x)| &= \left| \int_0^{+\infty} \xi(\inf_{z \in M(\theta_t \omega)} \|\varphi(t, \omega)y - z\|) - \right. \\ &\quad \left. \inf_{z \in M(\theta_t \omega)} \|\varphi(t, \omega)x - z\| \right) dt \Big| < \varepsilon. \end{aligned} \quad (3.29)$$

For $\|x - y\|$ sufficiently small, the points x and y belong to $A_M(\omega)$ and therefore there exists $T > 0$ such that

$$\int_T^{+\infty} \xi(\inf_{z \in M(\theta_t \omega)} \|\varphi(t, \omega)y - z\|) dt < \frac{\varepsilon}{2}, \quad (3.30)$$

$$\int_T^{+\infty} \xi(\inf_{z \in M(\theta_t \omega)} \|\varphi(t, \omega)x - z\|) dt < \frac{\varepsilon}{2}, \quad (3.31)$$

these inequalities follow from the convergence of the integral (3.23). Now, since an (θ, φ) of the RDS is continuous, given δ_2 we can find δ_3 such that

$$\|\varphi(t, \omega, y) - \varphi(t, \omega, x)\| < \delta_2(\omega) \text{ for } (x, y) \text{ and } 0 \leq t \leq T. \quad (3.32)$$

Since $\xi(\inf_{z \in M(\theta_t \omega)} \|x - z\|) = \mathcal{M}(\omega, x)$, since $\mathcal{M}(\omega, x)$ is continuous, consequently, for $\delta_2 > 0$, we have

$$|\xi(\inf_{z \in M(\theta_t \omega)} \|\varphi(t, \omega)x - z\|) - \xi(\inf_{z \in M(\theta_t \omega)} \|\varphi(t, \omega)y - z\|)| < \varepsilon/2T, \quad 0 \leq t \leq T. \quad (3.33)$$

Thus, for $\|x - y\| < \delta_3$ we have

$$|\mathcal{L}_1^*(\omega, x) - \mathcal{L}_1^*(\omega, y)| < \varepsilon, \quad (3.34)$$

and the properties (3.1)- (3.5) of the theorem have therefore been proved. We

shall now prove that also the property (3.6) is satisfied. For that take $\sigma_1 > 0$.

Let $T > 0$ and $\alpha > 0$ be such that $\inf_{z \in M(\theta_t \omega)} \|\varphi(t, \omega)x - z\| > \alpha$. For $0 \leq t \leq T$, and $\inf_{z \in M(\theta_t \omega)} \|x - z\| > \sigma_1$. Then there is $\beta > 0$ with $\xi(\inf_{z \in M(\theta_t \omega)} \|\varphi(t, \omega)x - z\|) \geq \beta$ for $0 \leq t \leq T$, hence

$$\mathcal{L}(\omega, x) \leq \exp \left[- \int_0^T \beta dt \right] - 1 < -\gamma_2 \quad (3.35)$$

which concludes the proof of the property (3.6).

It remains to be proved that also the property (3.7) is true. For that let $x \in A_M(\omega)$ and consider a sequence $\{x_n\} \subset A(M)$ with $x_n \rightarrow x$. Let $\delta_1 > 0$ be such that $H(M(\theta_t \omega), \delta_1(\omega)) \subset A(M)$ and $\{\tau_n\} \subset \mathbb{R}^+$ such that $\varphi(t, \omega, x_n) \notin S(M(\theta_t \omega), \delta_1(\omega))$ for $0 \leq t \leq \tau_n$, $\varphi(\tau_n, \omega, x_n) \in H(M(\theta_t \omega), \delta_1(\omega))$. Such a sequence $\{\tau_n\}$ exists and, in addition $\tau_n \rightarrow +\infty$ if not $x \in A_M(\omega) \cap \partial A_M(\omega)$, which is absurd. Let $\varphi(\tau_n, \omega, x_n) \rightarrow x^* \in H(M(\theta_t \omega), \delta_1(\omega))$.

As a result of the property (3.3) and the continuity of $\mathcal{L}(\omega, x)$, yield $-1 < \mathcal{L}(\omega, x^*) < 0$. From the property (iv) it then follows that there exists a real number, $\alpha > 0$ such that $\xi(\inf_{z \in M(\theta_t \omega)} \|\varphi(t, \omega, x_n) - z\|) \geq \alpha$ for $0 \leq t \leq \tau_n$. From the equality (3.20) it then follows that

$$= 1 + \mathcal{L}1 + \mathcal{L}(\theta_t \omega, \varphi(\tau_n, \omega, x_n)) \exp \left[- \int_0^{\tau_n} \xi(\inf_{z \in M(\theta_t \omega)} \|\varphi(t, \omega, x_n) - z\|) dt \right] (\omega, x_n). \quad (3.36)$$

Now the limit of the right-hand side of the (3.34) exists and it is equal to zero, hence $\mathcal{L}(\omega, x_n) \rightarrow -1$. The same reasoning applies if $\{x^*\} \subset A_M(\omega)$ and $\|x_n\| \rightarrow +\infty$, which concludes the property (3.7) and of the proof of the theorem is finished. ■

In order to illustrate the concept of a globally asymptotically stable random compact set, we will borrow the following example from Arnold and Schmalfuss [2].

Example

(Affine Random Dynamical Systems)

We will study the affine SDE in $\mathbb{R}^d$ given by

$$dx = \sum_{j=0}^m (A_j x + b_j) \circ dW^j, \quad \circ dW^0 := dt \quad (4.1)$$

where $A_j \in \mathbb{R}^{d \times d}$ and $b_j \in \mathbb{R}^d$, $j = 0, \dots, m$. (4.1) creates an affine RDS φ characterized by

$$\varphi(t, \omega)x = \Phi(t, \omega) \left(x + \sum_{j=0}^m \int_0^t \Phi(s, \omega)^{-1} b_j \circ dW_s^j(\omega) \right), \quad (4.2)$$

where Φ is the linear RDS created by the corresponding linear SDE

$$dx = \sum_{j=0}^m A_j x \circ dW^j.$$

Assume that

$$\lambda := \lim_{t \rightarrow \infty} \frac{1}{t} \ln \|\Phi(t, \omega)\| < 0.$$

Then

$$x_0(\omega) := \sum_{j=0}^m \int_{-\infty}^0 \Phi(t, \omega)^{-1} b_j \circ dW_t^j(\omega)$$

is the unique fixed solution of φ , i.e. $\varphi(t, \omega, x_0(\omega)) = x_0(\theta_t \omega)$. Then $A(\omega) = \{x_0(\omega)\}$ is a φ –invariant compact random set. It since

$$\lim_{t \rightarrow \infty} \|\varphi(t, \omega, X(\omega)) - x_0(\theta_t \omega)\| = \lim_{t \rightarrow \infty} \|\Phi(t, \omega)(X(\omega) - x_0(\omega))\| = 0$$

forever random variable X , then $A(\omega) = \{x_0(\omega)\}$ is an attractor and exponentially fast due to $\lambda < 0$.

The random set $A(\omega) = \{x_0(\omega)\}$ is stable according to the Lyapunov function $\mathcal{L}: \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^+$ given by

$$\mathcal{L}(\omega, x) := \|x_0(\omega) - x\|_{\kappa, \omega},$$

where $\|\cdot\|_{\kappa, \omega}$ is a random Lyapunov norm (see [1, 4.3]) so that

$$\|\Phi(t, \omega)\|_{\kappa, \omega, \theta_t \omega} \leq e^{(\lambda + \kappa)t}, t \geq 0, \kappa > 0 \text{ with } \lambda + \kappa < 0.$$

The $\mathcal{L}$ satisfies the axioms of Definition 3.1. Note that, by (4.3)

$$\begin{aligned} \mathcal{L}(\theta_t \omega, \varphi(t, \omega, x)) &= \|\Phi(t, \omega)(x_0(\omega) - x)\|_{\kappa, \omega} \\ &\leq \|\Phi(t, \omega)\|_{\kappa, \omega, \theta_t \omega} \mathcal{L}(\omega, x) \\ &\leq e^{(\lambda + \kappa)t} \mathcal{L}(\omega, x) < \mathcal{L}(\omega, x) \text{ for all } t > 0. \end{aligned}$$

Consequently, $\mathcal{L}$ is a Lyapunov function for $A(\omega)$, hence $A(\omega) = \{x_0(\omega)\}$ is globally asymptotically stable by Theorem 3.6.

CONCLUSION

The most significant finding of our research is that the existence of a Lyapunov function defined on compact random sets is a sufficient and necessary condition for the asymptotically stable stability of those sets. Furthermore, topological conjugacy between random dynamical systems maintains this kind of stability. Additionally, we proved that compact random sets can be contained inside an attraction region. Finally, we conclude that the affine RDS generated by affine SDE is globally asymptotically stable.

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